MECHANICS OF MATERIALS

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Mechanics of

MATERIALS

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MECHANICS OF MATERIALS

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Preface to the Second Edition

In the second edition of "Mechanics of Materials," no change has been made in the presentation or the topics so ably covered by Professor George and Professor Rettger.

The major change in arrangement consists in altering the order of the last four chapters, thereby bringing all the theory on beams into sequence. The data in all but four of the arithmetical problems have been changed and the rivet and column specifications have been brought up to date. A new feature of this edition is the article on the graphical solution of combined stresses. Many of the problems have been rewritten, some have been omitted, and two hundred fifty six review problems, without answers, have been added at the end.

The author here takes the opportunity to express his appreciation of the interest of teachers and others in the revision of this book as shown by the many helpful comments and criticisms offered.

E. V. HOWELL.

ITHACA, N.Y.,
February, 1943.

Preface to the First Edition

This text covers the essential topics of a first course in mechanics of materials and is intended primarily for the use of students of engineering in American universities.

Many years of experience have convinced the authors that the necessary parts of a course in mechanics of materials can be grasped most effectively if the treatment of the topics is simple and complete. References are, however, given in the text to more advanced treatises on the general subject. The addition of numerous complete examples, presented in as prominent a way as the theory of the subject matter itself, is considered by the authors to be a valuable and essential feature of the book. The limitation of the number of problems to what a reader can be reasonably expected to master in the time usually available for the subject has also been a prime consideration. Other problems based upon the text, the examples, and the given problems will suggest themselves to the reader. Both the examples and the problems have been devised to emphasize the fundamental theory underlying the solutions. In the selection of problems, the authors were guided by their conviction that a student gains more mastery of the subject if he can apply the basic theory to problems of some magnitude and difficulty. He profits less by the solution of many simple and elementary problems all resembling one another.

This text contains more material than is usually covered in an elementary course. The arrangement of the advanced material is such that any or all of the advanced topics may be eliminated without loss of sequence in the more elementary portions.

No attempt has been made to avoid the use of elementary calculus since the simplest and most concise treatment of many of the topics requires the calculus method. On the other hand, the usual repetition of the discussion of centroids and moment of inertia has been omitted on the assumption that this theory is given in mathematics or in a previous course in analytical mechanics.

In topics having to do with the design of members, modern methods of analysis and of treatment have been used. This is particularly true in the case of beams, columns, and members subjected to combined bending and column action.

One of the distinctive features of this text is the inclusion of the material in Chap. XII on the slope-deflection method, the value of which has been proved by the authors in many years of use. It has been found that the treatment of the slope-deflection method and the theorem of three moments, arouses the interest of students to a marked degree. At the same time it provides a valuable review of the theory of beams at an advantageous time near the end of the course. No difficulty has been experienced by students in mastering this subject. In one form or another, the slope-deflection method is used in more advanced methods of structural analysis, and the discussion given makes a useful introduction.

THE AUTHORS.

CORNELL UNIVERSITY,
March, 1935.

Contents

Greek Alphabet, Mathematical Data

Note.—Letters of the Greek alphabet will be used frequently. It is necessary for the reader to recognize these letters when they appear in the text and to be able to write them legibly.

GREEK ALPHABET

Letters	Names	Letters	Names
Α α	Alpha	Ν ν	Nu
Β β	Beta	Ξ ξ	Xi
Γ γ	Gamma	Ο o	Omicron
Δ δ	Delta	Π π	Pi
Ε ϵ	Epsilon	Ρ ρ	Rho
Ζ ζ	Zeta	Σ σ	Sigma
Η η	Eta	Τ τ	Tau
Θ θ	Theta	Υ υ	Upsilon
Ι ι	Iōta	Φ ϕ	Phi
Κ κ	Kappa	Χ χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
Μ μ	Mu	Ω ω	Omega

NUMERICAL CONSTANTS

$\pi = 3.1416$, or approximately $3\frac{1}{7}$, *i.e.*, $\frac{22}{7}$.

$\frac{1}{\pi} = 0.3183;\ \pi^2 = 9.870;\ \frac{1}{\pi^2} = 0.10132;\ \sqrt{\pi} = 1.772$

$1° = 0.01745$ radian. 1 radian $= 57.3°$.

$\sin 1' = \tan 1' = 0.000291$.

$\log_e N = 2.303 \log_{10} N$. $\log_{10} N = 0.4343 \log_e N$.

TRIGONOMETRY

$$\sin^2 A + \cos^2 A = 1.$$

$$\sin 2A = 2 \sin A \cos A.$$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A. \\ &= 2\cos^2 A - 1. \\ &= 1 - 2\sin^2 A. \end{aligned}$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}.$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}.$$

$$\tan^2 A = \sec^2 A - 1.$$

DIFFERENTIAL FORMS

$$\frac{d}{dx}(cx) = c.$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

$$\frac{d}{dx}x^n = nx^{n-1}.$$

$$\frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}\log_e x = \frac{1}{x}.$$

$$\frac{d}{dx}\sin u = \cos u\frac{dx}{du}.$$

$$\frac{d}{dx}\sin x = \cos x.$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}.$$

$$\frac{d}{dx}\cos x = -\sin x.$$

INTEGRAL FORMS

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

$$\int x^{-1}dx, \text{ or } \int\frac{dx}{x}, = \log_e x.$$

$$\int \sin\theta\, d\theta = -\cos\theta.$$

$$\int \cos\theta\, d\theta = +\sin\theta.$$

$$\int \sin\theta\cos\theta\, d\theta = \tfrac{1}{2}\sin^2\theta.$$

$$\int \theta\cos\theta\, d\theta = \theta\sin\theta + \cos\theta.$$

$$\int \sin^2\theta\, d\theta = \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta.$$

$$\int \cos^2\theta\, d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta.$$

$$\int e^x dx = e^x.$$

$$\int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.$$

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}.$$

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \log_e\left(x + \sqrt{x^2+a^2}\right).$$

$$\int\frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}}\sin^{-1}\left(\frac{2cx-b}{\sqrt{b^2+4ac}}\right), \text{ if } c > 0.$$

MECHANICS OF MATERIALS

CHAPTER I

STRESS AND STRAIN

FUNDAMENTAL CONCEPTIONS

1. Knowledge of Materials in Connection with Engineering Design.—Before entering upon the treatment of the topics which form the content of this book, we shall consider briefly in what way the subject matter is related to the important field of engineering design.

Suppose, as a first case, that an existing structure is to be the subject of an engineering investigation. It may have been weakened by decay or corrosion, or a change in the forces or loads to which the structure is to be subjected may be proposed and it may be necessary to reinforce its members, or to make some improvement in the arrangement or sizes of its parts.

It is evident that for such a structure the loads may be easily ascertained. The materials of which the parts are made may be examined. The essential dimensions of every member may be measured. The manner in which each part is acted upon by external forces and by forces coming upon it from adjacent parts may be determined. When all the necessary preliminary information is at hand, the investigation narrows down to the study of each separate member and terminates in the calculation of the strength (and possibly also of the deformation) of the member under the conditions in which it is found in the structure.

The second type of problem is of more frequent occurrence and usually is more interesting. The structure is at first only a creation of the imagination of the designer. Naturally the structure which he has in mind is likely to be influenced by former designs and by the ideas of others. Still a great deal of latitude

is offered to the designer of any new structure in the use of his inventiveness, his knowledge of facts and of materials, and his skill in analysis before his concept is fully developed.

Among the many considerations that the designer of a new structure must take into account, the following should be included:

a. The arrangement of members to form an advantageous type of structure.

b. The determination of the loads which the structure must bear.

c. The choice of materials to be used for its various parts.

d. The selection of suitable shapes for the members.

e. Provision for sufficient strength in every part of every member so that life and property may be safeguarded.

f. Calculations for possible excessive deformations of its members.

g. As a final step, the proposed structure should be compared with other conceivable and practicable structures on the basis of economy. Cost always is an important factor and often is the controlling factor.

Viewing the considerations stated above, the present text deals in an elementary way with the determination of the shape and dimensions of individual members so that a member may have the necessary strength and rigidity and at the same time will contain as little material as is practicable. That is, the question of economy will be limited to economy of material of individual members. Matters of general economy, shop practice, erection, and the like are merely mentioned or are left to specialized courses.

2. Mechanical Properties of Materials.—Experience has led to the conclusion that different materials are not affected in the same degree by the application of forces of a definite sort. A piece of wood may be bent much more easily than a bar of steel of the same size and shape. Such information is, however, not sufficiently definite to be of high engineering value. Mere casual observations of the behavior of materials under force action are naturally qualitative in character. It is clear that quantitative values, expressed in terms of established units, are needed if reliable laws are to be deduced concerning the behavior of pieces of material under the action of forces.

The foregoing statement explains why so much effort has been expended in obtaining and in correlating information on the

mechanical properties of various substances useful in construction. Most of such information is obtained from tests performed in laboratories that are especially equipped for testing operations. In addition, other facts have been gradually collected by the engineering profession as a result of experience in the best laboratory of all, the laboratory of actual use.

3. Mechanics of materials treats particularly of the internal stresses and also of the deformations of elastic solids due to the effect of external forces. This branch of mechanics is also called *resistance of materials;* or *strength of materials.*

Starting with experimentally determined mechanical properties of materials, laws and formulas are established by a process of reasoning that is mainly mathematical in character—laws and formulas that will be used in the analysis of stresses, in the design of members of a structure, and in the calculation of the changes of form of the members as caused by these stresses.

4. Stress.—It is a law of nature, first clearly formulated by Sir Isaac Newton, that forces always occur in pairs, *i.e.*, as

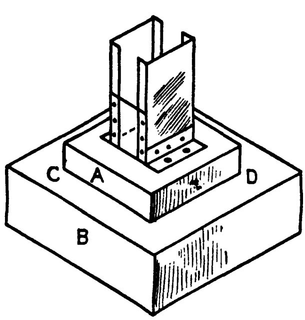


FIG. 1.

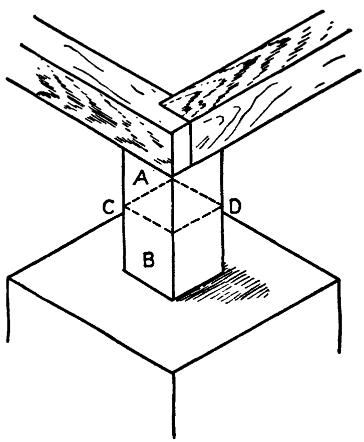


FIG. 2.

"action" and "reaction" between two bodies or between two parts of the same body. The forces constituting such a pair are always equal but are oppositely directed. In Mechanics of Materials, either force of such a pair is termed a *stress.*

In Fig. 1, A and B are two bodies in contact. The force exerted by A on B (or by B on A) is called the *stress* acting on their surface of contact CD. In like manner (Fig. 2), if a body is

imagined as divided into two parts A and B by a surface CD, the force exerted by A on B (or by B on A) is called the *internal stress* acting on their surface of contact.

The force exerted by A on B (or by B on A) is always distributed over their surface of contact. This surface may be small but it never can be a point or line. When the term stress is used, it should be with the understanding that it is "force distributed over a surface." No force should be termed a stress unless the surface over which it is distributed is clearly perceived. The surface may be curved or plane. A *plane surface* within a body will be called a *section*. Thus CD in Fig. 2 is a section.

5. In the same problem *a force may be treated as a stress* for one purpose and as a load or a reaction or a concentrated force for some different purpose.

Let OD be a beam resting on end supports and carrying a load G as shown (Fig. 3). Let CD be the surface of contact between the beam and the support N. The upward force F_1 exerted by the support against the beam,

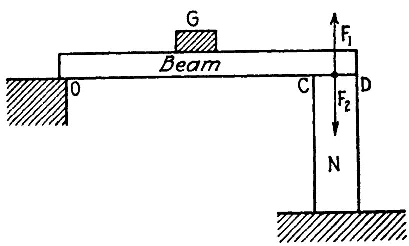


FIG. 3.

considered merely as a necessary force to keep the beam in equilibrium, becomes a reaction which for many purposes may be treated as a concentrated force. In reality this force is distributed over the surface CD of the beam, and if the force exerted on a unit area of this surface is too great the beam will fail (crumble or crush) at this surface. In like manner, for many purposes the equal and opposite force F_2 exerted by the beam on the support may be treated as a concentrated force. It is, however, actually distributed over the surface CD of the support, and if the force per unit area of this surface is too great the support will fail (crumble or crush) at this surface. Hence sufficient bearing surface must be provided between beam and support to avoid injury to either along their surface of contact. For this purpose it is necessary to consider the forces exerted between beam and support as distributed over their surface of contact. That is, it is necessary to investigate the stress acting on the surface CD.

6. Total Stress.—A stress is a force distributed over a surface. The resultant of this stress is called the *total stress* acting on that surface.

The total stress on a surface CD within a body in equilibrium may be found from the consideration that, if this surface is imagined as dividing the body into two parts A and B (Fig. 4), the resultant of the forces exerted by B on A together with all other forces acting on A form a system of forces in equilibrium

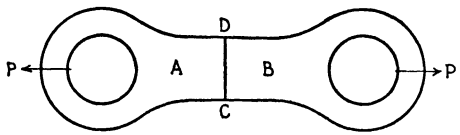


FIG. 4.

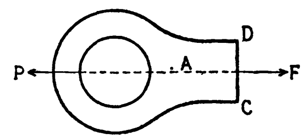
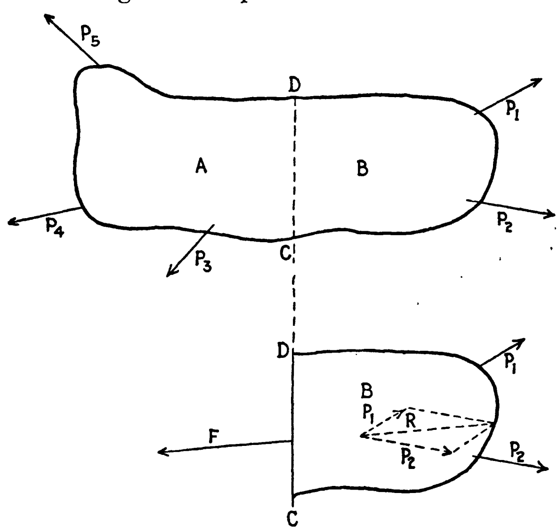


FIG. 5.

(Fig. 5). In like manner, the resultant of the forces exerted by A on B together with all other forces acting on B form a system of forces in equilibrium. That is, the total stress acting on CD may be found by considering the equilibrium of either A or B.

Example I.—Assume the eyebar (Fig. 4) to be in equilibrium. It is required to find the total stress F acting on the section CD. Imagine the plane CD to divide the bar into



FIGS. 6 and 7.

two parts A and B. Take A free (Fig. 5). If the weight of the bar is neglected, the only forces acting on A are P and F. For equilibrium to exist, these forces must be equal and opposite and must act in the same straight line; *i.e.*, $F = P$.

Example II. Fig. 6.—Assume the body in equilibrium under the action of the five forces P_1, P_2, P_3, P_4, P_5. It is required

to find the total stress F acting on the section CD. Take B free (Fig. 7). Now the forces acting on B are P_1, P_2, and F. The weight of the body is neglected. For equilibrium to exist, F must be equal and opposite to R, the resultant of P_1 and P_2; *i.e.*, $F = R$.

Note.—From Fig. 7, it is evident that the total stress on a section in a body need not act at right angles to the section nor need it act at the centroid of that section.*

7. Normal or Direct Stress. Shearing Stress.—The resultant stress, *i.e.*, the total stress on a section, may act normally, tangentially, or obliquely to the surface. If it acts obliquely, it may be resolved into a normal and a tangential component.

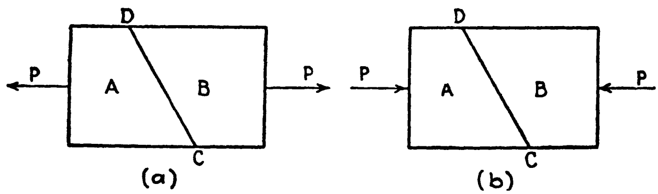


FIG. 8

For instance (Fig. 8), let a body be acted upon by two forces P and P. Imagine the section CD to divide the body into two parts A and B. Take A free (Fig. 9). From Art. 6 the total stress F acting on CD must be equal and opposite to P. Hence F acts obliquely to CD.

Let θ equal the angle F makes with the normal to the section CD. Resolving F into its normal and tangential components,

$$\left.\begin{aligned} F_n &= F \cos \theta, \\ F_s &= F \sin \theta. \end{aligned}\right\} \tag{1}$$

The two components of F, *i.e.*, F_n and F_s, usually are called the stresses acting on the section CD. The normal component F_n is called the *normal* or *direct stress*, and the tangential component F_s is called the *shearing stress* or the *shear*.

If the normal stress F_n acts away from the section (Fig. 9a), it is called a *tensile stress;* if it acts toward the section (Fig. 9b), it is called a *compressive stress*. That is, if the normal stress F_n on a section is tensile, the two parts A and B into which the body is imagined as divided by the section tend to pull apart; if compressive, the two parts push against each other.

* Center of gravity of that section.

Tensile and compressive stresses are essentially alike. If one is considered positive (+), the other is negative (−). When it is desired to distinguish between them, they may be denoted by F_t and F_c, respectively.

A compressive stress between two different bodies is called a *bearing stress.* For instance (Fig. 3, Art. 5), the compressive stress between the beam and the support N is a bearing stress.

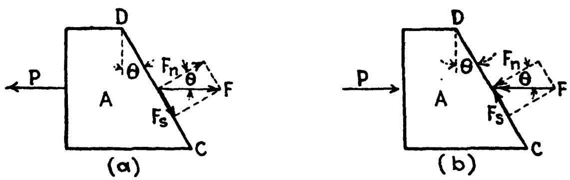


FIG. 9.

If we imagine a given section in a body to divide that body into two parts A and B, then the *shearing stress on that section measures the tendency of the two parts to slide one on the other along that section.* For instance, let the body represented by Fig. 8*b* be a block of ice. Evidently the two parts A and B tend to slide one on the other along the surface CD, and they would do so if the block of ice actually were cut along CD (Fig. 10). The block of ice, however, is not cut along CD and it is the shearing stress F_s which B exerts on A (Fig. 9*b*) that prevents A from sliding down on B along the surface CD. In like manner, it is the shearing stress which A exerts on B that prevents B from sliding up on A along the surface CD.

If the stress on a section acts wholly normally (no tangential component, *i.e.*, no shear on that section), it is called a *pure normal stress,* tensile or compressive as the case may be. For instance, with reference to Fig. 5 (Art. 6), the stress F on the right section CD acts wholly normally and hence F is a pure tensile stress (no shear along that section). In like manner, if the stress on a section acts wholly tangentially (no normal stress on that section), it is called a *pure shearing stress* or a *pure shear.*

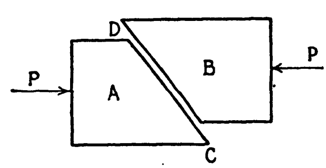


FIG. 10.

8. Unit Stress or Intensity of Stress.—If a stress on a plane surface is so distributed that equal amounts of stress act on all

equal areas—if, for instance, each square inch of surface bears the same amount of stress—the stress is said to be *uniformly distributed.* In the case of a uniformly distributed stress, the stress per unit area is called the *unit stress* or the *intensity of stress* and is found by dividing the total stress on a surface by the area of this surface. Thus, for a uniformly distributed stress, if F equals total stress and S equals intensity of stress (unit stress) acting on a surface of area A,

$$S = \frac{F}{A} \qquad \text{or} \qquad F = SA. \tag{2}$$

When the distribution of stress on a surface is not uniform, Eq. (2) gives the average intensity of stress. To find an expres-

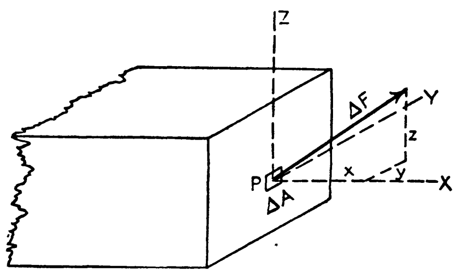


Fig. 11.

sion for the intensity of stress at a given point P of a surface, let ΔF be the total stress acting on an element of area ΔA surrounding the given point P (Fig. 11). The average intensity of stress on this increment of area is

$$S = \frac{\Delta F}{\Delta A}.$$

If ΔA is taken small enough, the stress on this increment of area may be considered as uniformly distributed and hence, in the limit as ΔA approaches zero, $\Delta F/\Delta A$ becomes the intensity of stress on an indefinitely small area surrounding the given point That is, the intensity of stress at the point P is

$$S = \lim_{\Delta A \doteq 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}. \tag{3}$$

Equations (2) and (3) apply to normal, shearing, and oblique stresses. In practice, however, they are applied as a rule only to normal and shearing stresses.

It is important to understand clearly what is meant by the *intensity of stress at a point.* Suppose that the intensity of stress (normal, shearing, or oblique) at a given point in a surface is S = 10,000 pounds per square inch. This means that, if a stress were distributed over a square inch of surface in the same way in which it is distributed over an indefinitely small area surrounding the given point, there would be a stress of 10,000 pounds uniformly distributed on this area of 1 square inch.

The unit stress or intensity of stress is sometimes called simply the "stress" when the meaning is evident from the context.

The English units used in expressing the intensity of stress are often written in abbreviated form. Thus "pounds per square inch" may be abbreviated into lb./sq. in., lb./in.^{2}, #/sq. in., or lb. per sq. in.; "tons per square foot," into tons/sq. ft. or tons per sq. ft., etc.

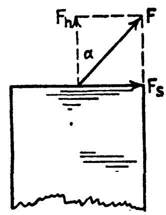


FIG. 12.

Example I.—If a stress of 30,000 lb. is uniformly distributed over a plane surface of 2 sq. in. (A = 2 sq. in.) and if the resultant stress (F = 30,000 lb.) makes an angle of 30° with the normal to this surface (Fig. 12), the unit normal stress is [Eq. (1)]

$$S_n = \frac{F_n}{A} = \frac{F \cos \alpha}{A} = \frac{30{,}000(0.866)}{2}$$
$$= 13{,}000 \text{ lb./sq. in.}$$

and the unit shearing stress is

$$S_s = \frac{F_s}{A} = \frac{F \sin \alpha}{A} = \frac{30{,}000(0.500)}{2}$$
$$= 7500 \text{ lb./sq. in.}$$

Example II.—If a normal stress of 80 lb. is distributed uniformly over an area of A = 0.005 sq. in., the normal stress per unit of area is S_n = 80/0.005 = 16,000 lb./sq. in. That is, if 16,000 lb. were uniformly distributed over an area of 1 sq. in., there would be 80 lb. on each bit of area measuring 0.005 sq. in.

9. Stress Solid.—The distribution of the normal or direct stress acting on a section (or a surface) is conveniently shown by representing the intensity of the normal stress at various points in the section. In Fig. 13, suppose it is desired to represent the

distribution of the normal stress acting on the section *CDEH* (section may have any shape but represented as rectangular for convenience). At the point *P* in the section draw an ordinate *PM* perpendicular to the section to represent to some scale the intensity of the normal stress at *P*. That is, $\overline{PM} = S$, where *S* equals intensity of the normal stress at *P*. Imagine this done for all points in the section. The solid *CDEH . . . IJKL*, thus formed, is called the *stress solid* for the section *CDEH*, its height at any point ($\perp$ section) representing the intensity of the normal stress at that point.

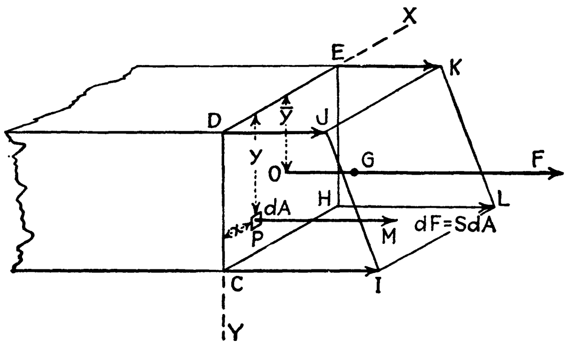


FIG. 13.

In the case of a shearing stress, the stress acts tangentially. If, however, the intensity of the shearing stress at each point in the section is represented also by an ordinate ($\perp$ section), a stress solid is constructed for the shearing stress. The distribution of the stress on a section may be fully represented, then, by drawing two stress solids—one for the normal or direct stress, and one for the shearing stress.

10. Theorems concerning the Stress Solid.—If Fig. 13 (Art. 9) is referred to, the stress (total) acting on the element of area dA surrounding the point *P* is $dF = SdA$ = volume of the elemental prism having dA as its base and *S* as its height; *i.e.*,

$$dF = dV \qquad \text{or} \qquad F = V. \tag{4}$$

Hence

I. *The total stress* (normal or shearing) *acting on a section equals the volume of the corresponding stress solid for that section.*

Point of Application of Resultant Stress.—It is frequently necessary to know at what point of a section the total stress

acts. To locate O, the point of application of the total stress acting on the section CDE (Fig. 13), take DE and DC as the X- and Y-axes, respectively. In elementary mechanics it was seen that the moment of the resultant force with respect to the X- or Y-axis equals Σ moments with respect to the same axis of the elementary forces dF acting on the elementary areas dA. That is,

$$\bar{x}F = \int x dF \qquad \text{and} \qquad \bar{y}F = \int y dF,$$

or

$$\bar{x} = \frac{\int x dF}{F} \qquad \text{and} \qquad \bar{y} = \frac{\int y dF}{F}.$$

Or, since $dF = dV$ and $F = V$ [Eq. (4)],

$$\bar{x} = \frac{\int x dV}{V} \qquad \text{and} \qquad \bar{y} = \frac{\int y dV}{V}. \tag{5}$$

Equations (5) are the calculus expressions for the x- and y-coordi-

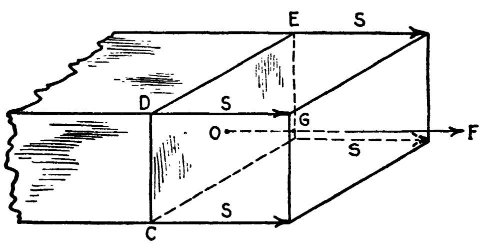


FIG. 14.

nates of the centroid (center of gravity) of the stress solid. Hence

II. *The total stress* (normal or shearing) *acting on a section passes through the centroid of the corresponding stress solid for that section.*

To illustrate (Fig. 13), if G is the centroid of the stress solid CK, then the total or resultant stress F acting on the section CE passes through G.

Stress Uniformly Distributed.—If the stress is uniformly distributed over a section, the stress solid becomes a right prism (Fig. 14). In the case of a right prism, a line drawn through the centroid of the prism and perpendicular to the base goes through the centroid of the base; *i.e.*, the point O is the centroid of the section CDE. Hence

III. *If a stress is uniformly distributed over a section, the total or resultant stress on that section acts at the centroid of that section.*

Note.—Theorem III is very important. In many cases the stress may be assumed as uniformly distributed over the section, and in such cases the total stress on that section may be represented as acting at the centroid of that section (Fig. 15).

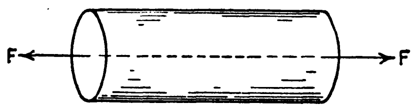


FIG. 15.

11. Stress Figures.—In many cases, the distribution of the stress on a section may be represented by a plane figure called the *stress figure*. Figure 16 shows the distribution of the stress on the section CD (a section of any shape), it being assumed that the intensity of stress is constant along any line perpendicular to the plane of the paper. That is, Fig. 16 may be considered a side view of the stress solid for that section.

An important case is that of a *rectangular* section with a *trapezoidally* distributed stress (Fig. 16). The intensity of stress along any line perpendicular to the plane of the paper is constant and the stress figure is a trapezoid. If h and b are the dimensions

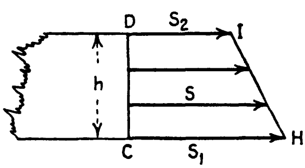


FIG. 16.

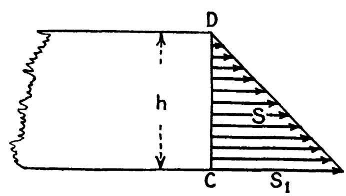


FIG. 17.

of the section ($b \perp$ plane of paper), and if S_1 and S_2 are the maximum and minimum unit stresses, respectively, then the volume of the stress solid is

$$V = hb\left(\frac{S_1 + S_2}{2}\right) = \text{area of section times average unit stress.}$$

If $S_2 = 0$, the stress figure becomes a triangle (Fig. 17) and the stress is said to be *triangularly* distributed. Hence, since $F = V$,

IV. *For a rectangular section with a trapezoidally or a triangularly distributed stress, the total stress equals the area of section multiplied by the average of the maximum and the minimum unit stress.*

Example. Fig. 18.—Find the total water pressure on the vertical face of a dam 40 ft. long if the water is 12 ft. deep.

In hydraulics it is shown that the intensity of the water pressure in pounds per square foot at a point y ft. below the water surface is

$$S = 62.5y. \tag{6}$$

Since S is directly proportional to y, the stress figure HCD is a triangle; *i.e.*, the pressure is triangularly distributed.

From Eq. (6) it follows that, at D, $S = 0$; and at C,

$$S = 62.5 \times 12 = 750 \text{ lb./sq. ft.}$$

By Theorem IV the total pressure on the dam is equal to the wetted surface times the average water pressure. Hence

$$P = (12 \times 40) \times \tfrac{750}{2} = 180{,}000 \text{ lb.}$$

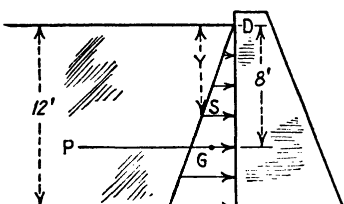


FIG. 18.

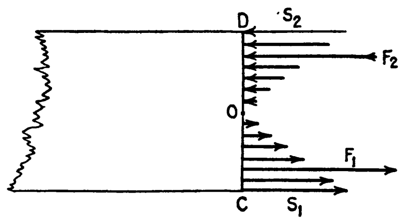


FIG. 19.

By Theorem II, P acts two-thirds the way down from D, or 8 ft. below D. (A line drawn through the center of gravity and parallel to the base of a triangle will cut the altitude h at a point $\frac{2}{3}h$ from the vertex.)

Note.—In many problems it is of importance to determine the actual distribution of stress on a section. In Fig. 19, assume that the stress on the lower part of the section CO is tensile, and that the stress on the upper part OD is compressive. This case occurs, as will be seen later, when a member is bent. If $F_1 = F_2$, the total stress on the section CD is $F = F_1 - F_2 = 0$. Although the total stress is zero (or in a similar case small), yet the intensity of the stress at C or D may be so great as to cause injury to the material.

In many problems it is useful to sketch the stress figure for a given section and to determine by aid of this figure the maximum intensity of the stress acting on the section. *The maximum*

intensity of stress must always be considered when determining the strength of a member.

12. Any straight piece of uniform (constant) cross-section is said to be *prismatic in form* and will be called a *prism*. Many of the standard structural shapes such as rods, bars, I-beams, and channels are prismatic in form.

The straight line joining the centroids of the end sections of a prism is the axis of the prism (Fig. 20). Thus, if M and N are the centroids of the end sections of a prism (not necessarily a rectangular prism), the straight line MN is the axis and goes through the centroid of every section of the prism.

A section made at right angles to the axis is a *right* or *transverse section*, and one cutting obliquely across the axis is an *oblique section*. In Fig. 20, CD is a right section and EF is an oblique section.

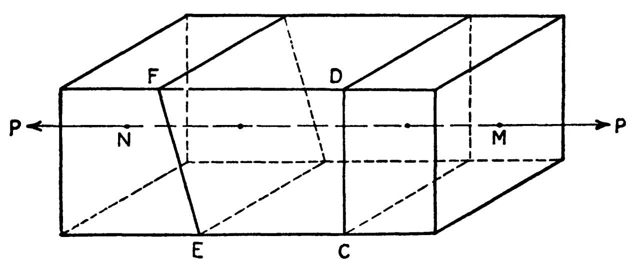


FIG. 20.

A concentrated force that acts along the axis of the prism, or a distributed force whose resultant acts along the axis of the prism, is called an *axial force* and is said to *act axially*. Thus (Fig. 20) P is such a force.

Note.—By definition, a prism is a straight piece. If forces are applied to this piece, it may be deformed (bent or twisted, etc.) so that it will no longer be exactly a prism. It will often be convenient to call it a prism, however, even after it is deformed. When this is done, it is meant to imply *that originally before the forces were applied it was a prism* and that it has changed but slightly from that shape.

13. Simple Tension and Compression.—The simplest state of stress occurs when a prism is acted upon by two equal and opposite axial forces, one at each end of the piece. In Fig. 21, let the prism be a tie-rod of the usual form and let CD be a right section. Take M free (Fig. 21*b*). Let F be the resultant

stress acting on the section CD. For equilibrium to exist, P and F must be equal and opposite and must act in the same straight line. By assumption, P is an axial force. Hence F is an axial stress and therefore acts normally to the right section.

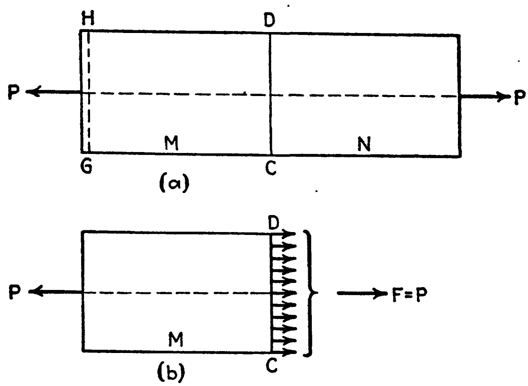


FIG. 21.

If P is a concentrated axial force (as represented in Fig. 21 or Fig. 22) or the resultant of a nonuniformly distributed axial force, the stress on an end section or on a section near the end (such as GH) is not uniformly distributed. However, if the section CD is taken far enough away from the ends, experience warrants the assumption that the stress becomes (very nearly, at least) uniformly distributed over this section. Hence the

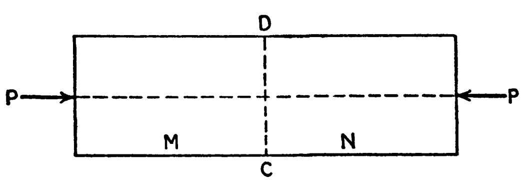


FIG 22.

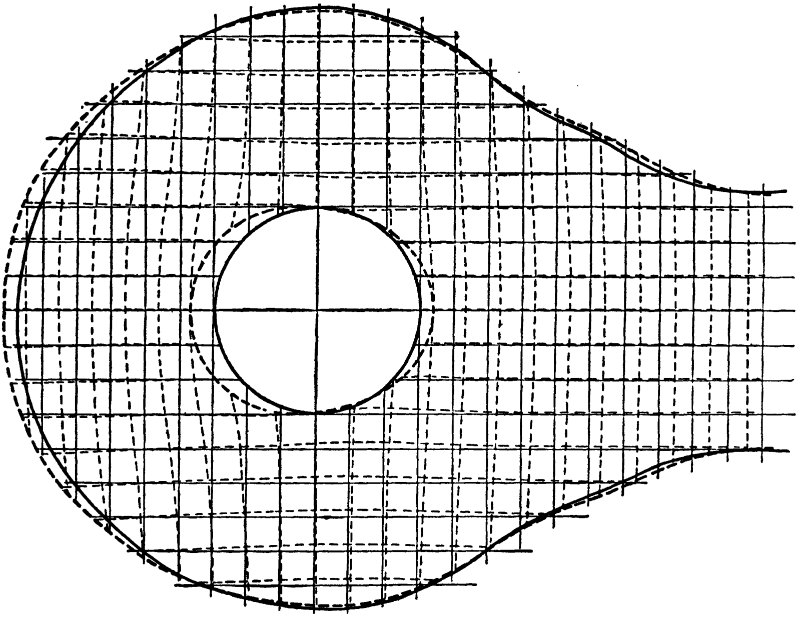
intensity of stress on a section far enough removed from the ends may be taken as

$$S = \frac{P}{A}$$

where A = area of section.

Figure 23 shows one end of an eyebar. The hole is called the "eye." The force is applied to the bar by means of a pin

going through the eye. This force is exerted on the bar along the left half of the eye (but not uniformly). Before the force was applied, lines were drawn on the bar in two directions dividing the surface of the bar into small areas (2 by 2 in.) as shown by the solid lines. When the force was applied, these small



TEST NO. 713
BAR No. 26. EYE A
Bar. 15 by 1⅝ in.
Elastic limit = 33,330 lb.
Ultimate strength = 65,220 lb.
Pin clearance = 0.051 in.

PERMANENT STRETCH OF PIN HOLE:
At 12,000 lb. = 0.000 in.
At 16,000 lb. = 0.002 in.
At 20,000 lb. = 0.004 in.
At 24,000 lb. = 0.012 in.
At rupture = 1.969 in.

FIG. 23.—New facts about eyebars. (*Theodore Cooper, Trans. A. S. C. E., vol.* 56, *p.* 411, 1906.)

areas became distorted as is shown by the dotted lines. It will be seen later that the distortion at any point in a body will depend upon the magnitude and the nature of the stress acting at that point. From the irregularity in the distortion of the small areas along a section through or near the eye (as shown by the dotted lines), it is evident that the stress along this section is not uniformly distributed. Note, however, that

at some distance to the right of the eye the dotted lines tend to become straight and parallel, indicating that the stress on a right section tends to become a uniformly distributed normal stress. *On a section whose distance from the center of the hole is about two diameters of the hole, the stress may be considered as uniformly distributed.*

A pure normal stress (no shear anywhere on that section) will be called a *simple normal stress* if it is uniformly distributed over that section. It may be compressive or tensile.

A member is said to be in *simple tension* or in *simple compression* if on every right section (not too near the point of application of an external force, and not too near a hole, notch, or sudden change in the section) the stress is a simple normal stress. Accordingly, a prism subjected to an axial load is in simple tension or compression. The state of stress at or near the ends of such a prism may require special consideration. For the present, assume that the load is uniformly distributed over the end, or that only that part of the prism is under consideration for which the stress on every right section is a simple normal stress.

Example I.—A steel rod of sectional area $A = \frac{1}{2}$ sq. in. is subjected to an axial pull of 5 tons. What unit stress does this imply on a right section of the rod?

Ans. $S = P/A = 5/0.5 = 10$ tons/sq. in. $= 20{,}000$ lb./sq. in.

Example II.—A rod is to be subjected to an axial pull of $P = 4000$ lb. If the maximum allowable stress on a right section of this rod is $S = 20{,}000$ lb./sq. in., what must be the sectional area of the rod?

Ans. $20{,}000 = 4000/A$; or $A = 0.20$ sq. in.

14. Intensity of Stress on an Oblique Section of a Prism under Simple Tension or Compression.—If a prism (Fig. 24*a*) is acted upon by two equal and opposite *axial* forces P and P, one at each end, the stress on a right section CD of this prism is a simple normal stress of intensity

$$S = \frac{P}{A},$$

where A = sectional area of the prism (Art. 13).

The question now arises whether the intensity of the stress on some oblique section may not be greater than that on a right section. Consider the portion of the prism between the right section CD and the oblique section HG (Fig. 24*b*). The total

stress on the section CD is $P = AS$. For equilibrium to exist, the stress on the oblique section HG must be such that its resultant must be equal and opposite to that on CD.

For convenience, replace the stresses on CD and HG by their resultants P and P (Fig. 24c). Now the stress on CD acts normally to CD but that on HG acts obliquely and makes an angle α with the normal to HG. Note that α is also the angle the oblique section makes with a right section. Resolve the

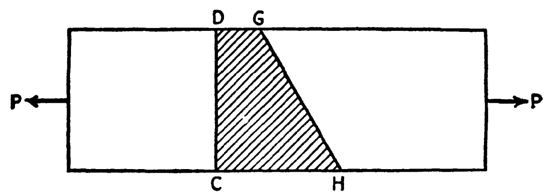


FIG. 24a.

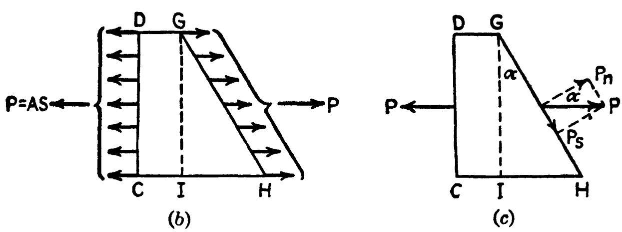


FIG. 24b and c.

resultant stress P acting on HG into its normal and tangential components P_n and P_s, *i.e.*,

$$P_n = P \cos \alpha \qquad \text{and} \qquad P_s = P \sin \alpha.$$

If A' equals the area of oblique section HG, then $A' = A/\cos \alpha$, and the intensity of the normal stress on this section is

$$S_n = \frac{P_n}{A'} = \frac{P \cos \alpha}{\dfrac{A}{\cos \alpha}} = \frac{P}{A} \cos^2 \alpha.$$

Or, since $P/A = S$ equals the intensity of normal stress on the right section,

$$S_n = S \cos^2 \alpha.$$

In like manner, the intensity of the shearing stress on the oblique plane HG is

$$S_s = \frac{P_s}{A'} = \frac{P \sin \alpha}{\frac{A}{\cos \alpha}} = \frac{P \sin \alpha \cos \alpha}{A} = \frac{S \sin 2\alpha}{2}.$$

Hence, if S *equals the intensity of the normal stress on a right section of a prism under simple tension or compression, then on an oblique section making an angle* α *with the right section the intensities of stress are respectively,*

Normal stress:

$$S_n = S \cos^2 \alpha. \tag{7}$$

Shearing stress:

$$S_s = \frac{S \sin 2\alpha.}{2} \tag{8}$$

From Eq. (7) it follows that S_n is a maximum if $\cos^2 \alpha = 1$, *i.e.*, if $\alpha = 0$. From Eq. (8) it follows that S_s is a maximum if $\sin 2\alpha = 1$, *i.e.*, if $2\alpha = 90°$ or $\alpha = 45°$. Putting $\alpha = 45°$ in Eq. (8)

$$\text{Max. } S_s = \frac{S}{2}. \tag{9}$$

Hence, if a prism is in simple tension or compression,

1. *The intensity of the normal stress is a maximum on a right section.*

2. *The intensity of the shearing stress is a maximum on an oblique section making an angle of* 45° *with a right section and its magnitude is*

$$\text{Max. } S_s = \frac{S}{2},$$

where S = intensity of normal stress on a right section.

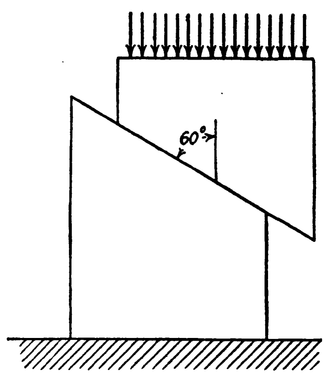


FIG. 25.

Example. Fig. 25.—In a block of wood whose sectional area was A = 16 sq. in., the grain made an angle of 60° with the axis of the block (30° with a right section). The prism was placed in simple compression and failed in shear along the grain when the total load was P = 29,600 lb. Required to find the corresponding intensity of the shearing resistance of the wood along the grain.

The intensity of the normal stress on a right section was

$$S = \frac{P}{A} = \frac{29,600}{16} = 1,850 \text{ lb./sq. in.}$$

Using Eq. (8), putting $\alpha = 30°(\sin 2\alpha = 0.866)$,

$$S_s = \frac{1{,}850 \times (0.866)}{2} = 800 \text{ lb./sq. in.}$$

That is, the ultimate strength of the wood in shear along the grain was 800 lb./sq. in.

Problem 1.—A steel bar, 0.8 in. in width and 0.16 in. in thickness, is tested to failure by pulling it apart with an ultimate load of 8950 lb. Failure occurred in shear on a plane making an angle of 42° with the cross-section. (*a*) Find the unit shearing stress on this plane. (*b*) Compute also the theoretical maximum unit shearing stress.

Ans. (*a*) 34,800 lb./sq. in.; (*b*) 35,000 lb./sq. in.

15. Columns.—For a prism to be under simple compression it must not bend. As soon as it bends (Fig. 26*a*), the end forces P and P, although applied at the centroids of the end

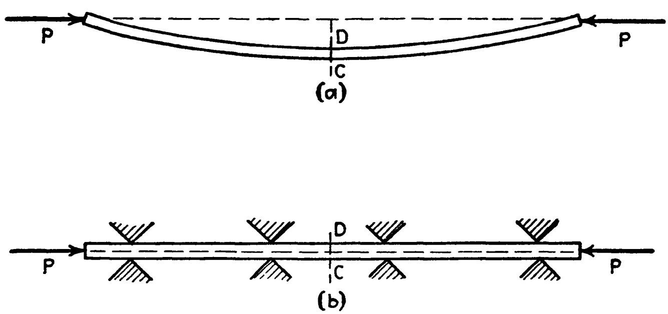


FIG. 26.

sections, are no longer axial forces since their line of action no longer goes through the centroid of every section such as CD. Hence the stress on a right section is no longer uniformly distributed and therefore is not a simple normal stress.

A prism that is relatively so long that it bends when two equal and opposite end forces are applied is called a *column*. Columns require special consideration and will be treated in a later chapter.

When reference is made to a prism under simple compression, it should be understood that the prism is well supported laterally so as to prevent bending or buckling (Fig. 26*b*) or that it is relatively so short that its tendency to bend may be neglected.

16. Simple Shear.—If a stress acts wholly tangentially to a section (normal stress zero) it is called a *pure shear*. If, in

addition, it is uniformly distributed over the section, it is called a *simple shear.*

A prism acted upon by two equal and opposite axial forces (one at each end) is a familiar example of simple tension or compression. It will be seen later that the state of simple shear can be brought about only by a combination of stresses. There are, however, familiar cases in which the stress on a section approximates that of simple shear.

As an example (Fig. 27) consider the action of two equal forces tending to push off toward the end of the timber the two blocks CD and $C'D'$. If P is the total force in the member, then by assumption the force on the right face of a block is $Q = P/2$. Consider the upper block CD. Owing to the force Q acting on its surface, a shearing resistance is developed along the surface CD. It is not possible to determine exactly how this resistance is distributed over the surface CD. It is customary to consider the average unit shear on this surface and to treat the shear as if uniformly distributed over the surface.

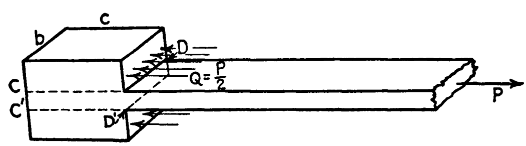


FIG. 27.

If A equals the area of the surface CD and S_s equals the (average) unit shear on this surface,

$$S_s = \frac{Q}{A} \quad \text{or} \quad Q = AS_s.$$

Since by assumption $Q = P/2$ where P equals the total force (pull) in the member,

$$S_s = \frac{P}{2A} \quad \text{or} \quad P = 2AS_s.$$

If CD is a rectangular area of dimensions b and c, then

$$S_s = \frac{P}{2bc} \quad \text{or} \quad P = 2bcS_s.$$

In Fig. 28 is shown one end of a wooden roof truss. The strut or member M is set into a notch in the bottom chord N. The portion BDE of the bottom chord prevents the lower end of M from slipping or sliding. Owing to the horizontal component

of P, a shearing stress is induced along the surface BD whose average intensity is

$$S_s = \frac{P \cos \alpha}{A} \tag{10}$$

where A = area of section BD.

Example I.—Referring to Fig. 28, let $P = 3$ tons, $\alpha = 30°$, and $b = 6$ in. What must be the length a of the surface BD

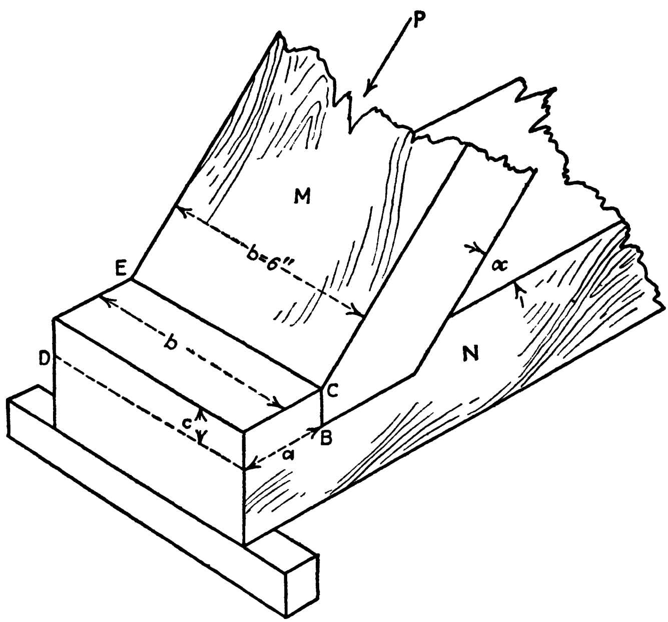


Fig. 28.

if the average unit shearing stress on this surface is not to exceed $S_s = 100$ lb./sq. in.?

$$100 = \frac{6000 \times (0.866)}{a \times 6} \qquad \text{or} \qquad a = 8.66 \text{ in.}$$

Example II.—The bearing stress (compressive stress) between M and N (Fig. 28) along their surface of contact BE must be investigated. What must be the depth c of the notch if the average bearing stress on the surface BE is not to exceed $S_c = 1000$ lb./sq. in.? Given $b = 6$ in., $\alpha = 30°$, and $P = 3$ tons.

$$1000 = \frac{6000 \times (0.866)}{6 \times c} \qquad \text{or} \qquad c = 0.866 \text{ in.}$$

Example III. Fig. 29.—The ends of a steel I-beam rest on concrete piers. The beam carries a central load of 6 tons. Required to find the necessary bearing surface at the piers if $S_c = 400$ lb./sq. in. is the safe unit bearing stress for the piers. Concrete is much weaker in its resistance to compression than steel. Hence only the bearing stress on the piers needs be investigated. Bearing on each pier equals one-half of the load, equals 3 tons, or 6000 lb.
Therefore

$$400 = \frac{6000}{A} \qquad \text{or} \qquad A = 15 \text{ sq. in.}$$

If the flange of the beam is 5 in. wide and the beam is to rest directly on the piers, then the beam must rest on each pier over a length of 3 in.

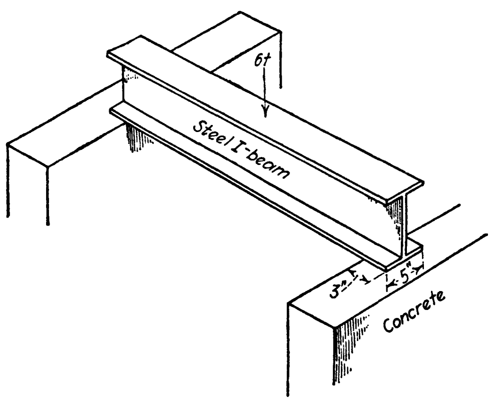


Fig. 29.

17. Strain.—When a body is subjected to force it will change its shape. This change of shape is called *strain* or *deformation.**

In theoretical mechanics bodies, as a rule, are considered rigid, *i.e.*, as incapable of changing their shape or form. This assumption is permissible since the changes in the dimensions of a body due to the forces acting on it are, as a rule, relatively so small that, in comparison with the original dimensions of the

* In nontechnical literature the word "strain" sometimes is used to designate force or stress. For example, such expressions as "the rope was not able to stand the strain" are sometimes used. In modern technical mechanics, however, strain applies to the change in shape of a body produced by stress.

body, these changes may be neglected for practical purposes. When a structure, such as a bridge, for example, is loaded, its members change their shape. Some are elongated, some shortened, some bent, etc. In a properly designed structure, however, any alterations in distances and angles within the structure due to any change in form of the structure or of its parts are relatively so small that *for many purposes the main dimensions of a structure or of its parts after strain may be taken the same as those before strain.*

Illustration.—Figure 30 represents a steel rod subjected to a pull P. The rod is 15 ft. long and has a sectional area of $A = \frac{1}{2}$ sq. in. If the maximum allowable unit stress is $S = 16{,}000$ lb./sq. in., then the maximum allowable pull is $P = AS = 8000$ lb. It will be seen later that, owing to this pull, the rod will elongate (stretch) about 0.096 in., or 0.008 ft. However, when compared with the original length of the rod, the elongation is so small

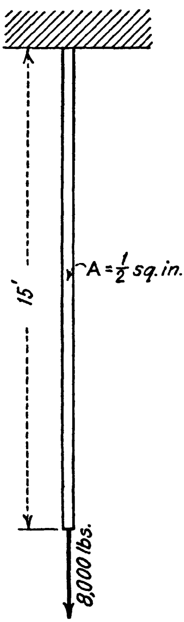


FIG. 30.

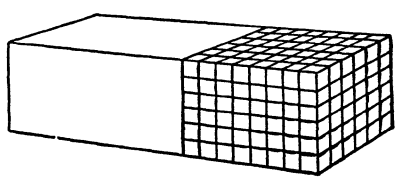


FIG. 31.

that for practical purposes the length of the rod after strain may be taken the same as that before strain, *i.e.*, as $L = 15$ ft.

18. Elemental Prism, or Element.—Frequently it is found convenient to think of a body as composed of very small prisms in the form of right parallelepipeds. One of these small blocks is called an *elemental prism* or an *element.* In Fig. 31, we imagine the body divided into elemental prisms (rectangular blocks) by three sets of planes, the three sets of planes being at right angles to each other. (A similar method of marking areas on the surface of an eyebar was used in Art. 13.)

Frequently, the elemental prism may be thought of as cubical in form, *i.e.*, as a *cubical element* (Fig. 32). Evidently, the cube may be represented by its front face, *ABCD.* The edge of the

elemental cube may be designated by a. It should be remembered, however, that the cube is infinitesimal in size so that a equals dx or dy or dz, as the case may be.

The advantages of dealing with elemental prisms often are very great. A body subjected to a system of forces may be elongated in one direction and compressed in another direction. At the same time, the body also may be bent and twisted. As a result, the stress on a section of this body may be distributed in a very complicated manner. If, however, an elemental prism is considered, the prism being infinitesimal in size, the *stresses on its faces may be assumed as uniformly distributed,* and

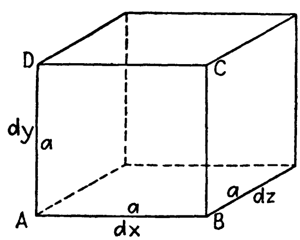


FIG. 32.

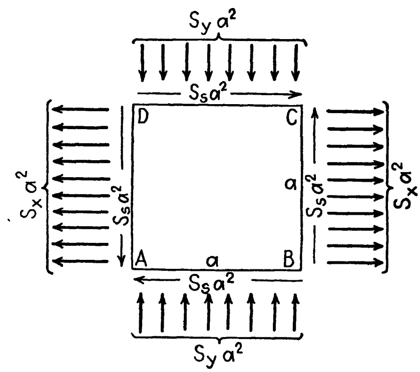


FIG. 33.

those on opposite faces as equal and *oppositely directed.* If not exactly equal, any pair of stresses such as the normal stresses on a pair of opposite faces can differ at most by an infinitesimal of a higher order and therefore, in the limit as the dimensions of the prism approach zero, these stresses approach equality.

For instance (Fig. 33), the elemental cube is represented as acted upon by three sets of stresses, *i.e.*, by two uniformly distributed tensile stresses each equal to S_xa^2 (S_x = intensity of stress), two uniformly distributed compressive stresses each equal to S_ya^2, and four (uniformly distributed) shearing stresses each equal to S_sa^2.

Note that the four shearing stresses are represented as being all of the same intensity S_s, and that those on two adjacent faces act either both toward or both away from the corner common to the two faces. In the next article it will be shown that this is always the case for an elemental prism.

19. Shearing Stresses on the Four Faces of an Elemental Prism Are of Equal Intensity.—With reference to Fig. 33 (Art. 18) it should be noted that the normal stresses on any pair of opposite faces of an elemental prism are equal and opposite and therefore form a system of forces in equilibrium. That is, the normal stresses may be removed from the faces of an elemental prism without affecting its equilibrium. The shearing stresses on the faces of an elemental prism, therefore, must be in equilibrium among themselves.

Consider now an elemental cube (Fig. 34) and assume that the intensities of the shearing stress on four of its faces are

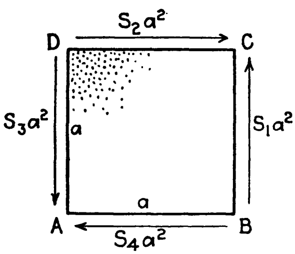


FIG. 34.

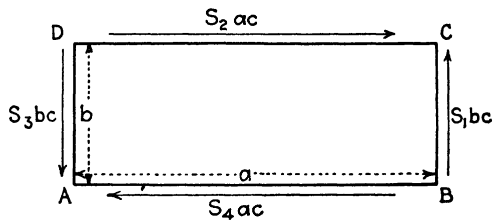


FIG. 34A.

S_1, S_2, S_3, and S_4, respectively. For equilibrium to exist, Σ moments about the edge A must equal zero. Or

$$S_1a^2 \times a - S_2a^2 \times a = 0.$$

Therefore

$$S_1 = S_2.$$

In like manner, by putting Σ moments about $B = 0$, $S_2 = S_3$. That is, *the intensities of the shearing stress on the faces of an elemental prism are always equal.* Moreover, equilibrium cannot exist unless the shearing stresses on any two adjacent faces are directed either both toward or both away from the edge common to the two faces. Thus both S_1 and S_2 must be directed either toward the edge C or away from C.

Note.—This theorem is true in general only for an elemental prism since it is only for an elemental prism that the normal stresses on the faces of the prism may be assumed to form a balanced system of forces which may be removed without affecting the equilibrium of the prism.*

* In the derivation of the theorem of this article an elemental cube was taken. Any elemental prism, however, may be taken. Let the elemental

20. Deformation of a Body Accounted for by Strains of Its Elements.—Consider now the *strain of an elemental prism* upon whose faces more than one set of stresses act (Fig. 33, Art. 18). Note that the prism is represented as acted upon by a set of (uniformly distributed) tensile stresses $S_x a^2$, a set of compressive stresses $S_y a^2$, and a set of shearing stresses $S_s a^2$. Owing to the stresses acting on its faces, the prism will be strained (deformed). The strain of this prism will be the combination of the strains produced by each set of stresses acting alone.* Hence, if the strain of the prism due to each set (acting alone) can be determined, the strain of the prism if all act together can be determined, and in many cases the deformation of the body from which this prism is taken can be determined. That is, the study of the deformation of a body rests in its last analysis upon the study

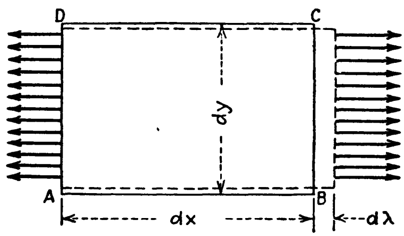


FIG. 35.

of the strains of an elemental prism of this body, the prism in turn being subjected to each set of stresses acting alone.

21. Unit Longitudinal Strain.—If an elemental prism is acted upon by *simple tensile stresses* on two opposite faces (Fig. 35), the strain of the prism will consist of an elongation (stretching) of the prism in the direction of the stress and a contraction of the prism at right angles to the direction of the stress. The edges AB and DC will be elongated and the edges AD and BC will be shortened. The elongation of the prism in the direction of the stress is called a *simple tensile strain* or a *simple strain of*

prism (Fig. 34A) have the dimensions a, b, and c ($c \perp$ plane of paper). If S_1 is the intensity of shearing stress on the face BC, the total shearing stress on BC is S_1bc. Similarly, the total stress on DC is S_2ac. Putting $\Sigma M_A = 0$,

$$(S_1bc) \times a - (S_2ac) \times b = 0.$$

Therefore

$$S_1 = S_2.$$

* Provided the elastic limit is not exceeded (Art. 25).

elongation, and the accompanying contraction of the prism at right angles to the stress is called the *lateral contraction*.

In like manner, if the prism is acted upon by *simple compressive stresses* on two opposite faces, the strain is a *simple compressive strain* (shortening in the direction of the stress) accompanied by a *lateral expansion*.

Tensile and compressive strains are essentially alike. One is an elongation of the prism and the other is a shortening of the prism in the direction of the stress. Both are conveniently called *longitudinal strains*. Similarly, lateral contractions and lateral expansions are essentially alike. Both may be called *lateral strains*.

Let dx equal the original length of prism (Fig. 35), and let $d\lambda$ equal the elongation (or contraction) of the prism in the direction of the stress. The unit longitudinal strain (strain per unit length of prism) is

$$\epsilon = \frac{d\lambda}{dx}.$$

If it is desired to distinguish between unit tensile strain and unit compressive strain, ϵ_t and ϵ_c may be used respectively to designate them.

In like manner (Fig. 35), if dy is a lateral dimension and $d\lambda'$ is the corresponding lateral strain, the *unit lateral strain* is

$$\epsilon' = \frac{d\lambda'}{dy}.$$

22. Poisson's Ratio.—Poisson is credited with the discovery that for any given material the ratio of unit lateral strain to unit longitudinal strain is a constant (approximately). This ratio is called *Poisson's ratio* and will be designated by m; *i.e.*,

$$m = \frac{\epsilon'}{\epsilon} = \text{constant}. \tag{11}$$

Hence, if Poisson's ratio is known, the lateral strain can be calculated in terms of the longitudinal strain.

The determination of Poisson's ratio requires very sensitive instruments. The following are approximate values: for iron, $m = 0.25$; for steel, $m = 0.27$.

23. Prism of Length L. Unit Longitudinal Strain.—If a prism of length L (a steel rod, for instance) is put under simple tension (or compression), the stress on any right section of the

prism is a uniformly distributed normal stress and the stresses on all right sections are equal. If the prism is assumed to be homogeneous, all equal lengths will be elongated (or shortened) equally, and the unit longitudinal strain (elongation or shortening per unit length) is

$$\epsilon = \frac{\lambda}{L} \tag{12}$$

when λ = elongation (or shortening) of a finite portion of this prism.

L = original length of this portion.

Hence, when the longitudinal strain of a homogeneous finite prism under simple tension or compression is studied, it will be found convenient to deal with a portion of finite length. Since ϵ is a length divided by a length, ϵ is an abstract number or mere ratio and any consistent unit of length may be used for λ and L.

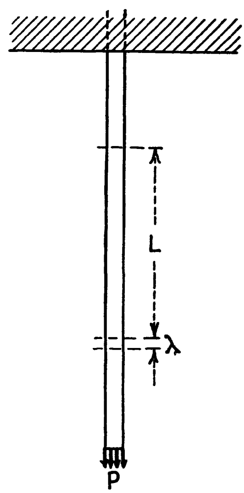


FIG. 36.

Example I.—A steel rod is subjected to an axial pull P (Fig. 36). A portion of length 4 ft. is elongated 0.0192 in. What is the unit tensile strain?

$$\epsilon = \frac{\lambda}{L} = \frac{0.0192}{4 \times 12} = 0.0004 \text{ (an abstract number).}$$

That is, every inch of the rod elongates 0.0004 in.; every foot, 0.0004 ft., etc. If the centimeter is used as the unit of measure, λ = 0.0488 cm. and L = 122 cm. With these values

$$\epsilon = \frac{0.0488}{122} = 0.0004$$

as before.

Example II.—Originally the rod of Example I had a diameter of d = 0.5 in. Find the lateral contraction. Take Poisson's ratio as m = 0.27.

From Eq. (11) of Art. 22, the unit lateral contraction is $\epsilon' = m\epsilon$. With ϵ = 0.0004 (Example I),

$$\epsilon' = 0.27 \times 0.0004 = 0.000108.$$

Hence the total lateral contraction of the rod is

$$\lambda' = \epsilon' d = 0.000108 \times 0.5 = 0.000054 \text{ in.}$$

and the final diameter of the rod is

$$d' = d - \lambda' = 0.499946 \text{ in.}$$

Note that the lateral contraction of the rod is so small that for practical purposes it may be neglected. In Advanced Mechanics, however, there are important problems whose solution necessitates the use of Poisson's ratio.

24. Unit Shearing Strain.—Let $ABCD$ (Fig. 37) represent an elemental prism in its unstrained state, and let $ABC'D'$ represent this prism when strained under the action of simple shearing stresses. Note that the angles at the four corners of the prism

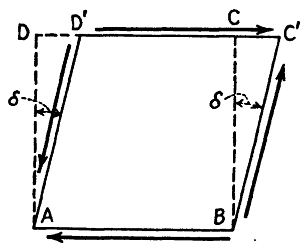


FIG. 37.

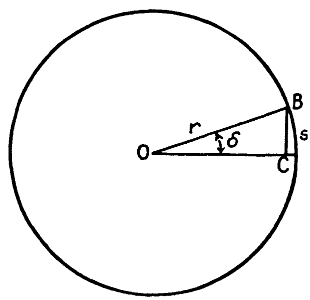


FIG. 38.

all change by the same amount δ; *i.e.*, those at A and C become smaller than 90° by an amount δ, and those at B and D become larger than 90° by an amount δ.

The angle δ is called the *angle of shear* or the *angular distortion.* The strain of an elemental prism due to the four simple shearing stresses acting on four faces is called a *simple shearing strain* and consists in a change of angles at the corners of the prism, all angles changing by the same amount δ.

If for convenience the face AB is assumed fixed, the strain consists of the sliding of the face DC relative to the face AB, the corner C moving from C to C'. The sliding of the face DC relative to AB (*i.e.*, CC') sometimes is called the *total shearing strain.* The unit shearing strain is defined as follows:

$$\textit{Unit shearing strain} = \frac{CC'}{BC} = \tan \delta.$$

Note that if BC is unity (unit prism) then CC' is the unit shearing strain.

The maximum allowable angle of shear δ for a material in a properly designed structure is so small that for $\tan \delta$ we may put δ measured in radians.

Hence

$$\text{Unit shearing strain} = \delta$$

where δ = angle of shear in radians.*

Note.—It should be remembered that unit strain (whether longitudinal or shearing) is an abstract number; *i.e.*, a unit strain always is a length divided by a length. Hence unit strain has no dimensions.

25. Elasticity. Elastic Limit. Set.—Elasticity is the name given to that property which most materials possess to a greater or less extent by virtue of which a body will recover its original form as soon as the forces producing a deformation are removed.

A quantitative study of the elastic properties of a material rests in its last analysis upon the study of the elastic properties of a prism of this material, the prism being subjected to one of the simple stresses (tensile, compressive, or shearing). Experiments show that the strain (deformation) of such a prism will disappear as soon as the stress is removed provided the unit stress does not exceed a certain value called the *elastic limit* of that stress for that material. If, however, the stress exceeds its elastic limit, the prism will not fully recover its original form when the stress is removed. That part of the deformation which remains when the stress is removed is called the permanent set, or simply the *set*. If a soft steel rod of length L = 10 ft. and of sectional area $A = \frac{1}{2}$ sq. in. is subjected to an axial pull of P = 15,000 lb., the maximum unit tensile stress developed in that rod is

* From trigonometry by definition (Fig. 38),

$$\delta \text{ in radians} = \frac{\text{arc}}{\text{radius}} = \frac{s}{r} = \text{abstract number.}$$

For a *very small angle*, $OC = r$ (approximately), and $BC = s$ (approximately). Therefore, for a very small angle we may put, δ being measured in radians,

$$\sin \delta = \frac{BC}{r} = \frac{s}{r} = \delta,$$

$$\tan \delta = \frac{BC}{OC} = \frac{s}{r} = \delta.$$

$$S = \frac{P}{A} = 30{,}000 \text{ lb./sq. in.}$$

Owing to this stress, the rod will elongate about $\lambda = 0.120$ in. When the stress is removed, the rod returns to its original length, $L = 10$ ft. Hence a unit stress of 30,000 lb./sq. in. is within the elastic limit for this rod in tension. If now the rod is subjected to an axial pull of $P = 20{,}000$ lb., the unit tensile stress is $S = P/A = 40{,}000$ lb./sq. in. The elongation of the rod will be about $\lambda = 0.240$ in. When the stress is removed, the rod will have a length of about 10 ft. 0.08 in. Hence there is a set of 0.08 in. That is, a unit stress of 40,000 lb./sq. in. is beyond the elastic limit for this rod in tension.

For each material there are three elastic limits—one for tension, one for compression, and one for shearing—*the elastic limit of any one stress being the maximum intensity of that stress that can be developed in the material without producing a permanent set.*

As soon as the stress in any member of a structure exceeds its elastic limit, that member is permanently deformed. Hence to avoid injury to the structure the stresses induced in its members should not exceed their elastic limits. This is particularly to be avoided in case of repeated, varying, and reversed loads.

Note.—Certain experiments of great precision indicate that perfect elasticity may not exist for any material. That is, when a body is deformed by the application of external forces, it never completely recovers as soon as the external forces are removed. This is obviously true for plastic material such as lead and tar. For iron, steel, wood, and some of the other more important engineering materials, so long as the material is not stressed beyond the so-called elastic limit, any failure of the material to recover completely when the stress is removed is so slight that it cannot be detected with the instruments ordinarily used. Practically, therefore, these materials possess an elastic limit.

26. Hooke's Law. Modulus of Elasticity—In its modernized form, Hooke's law states that within the elastic limit stress of any one kind is proportional to the strain it produces. This is equivalent to stating that the *ratio of unit stress to the unit strain it produces is a constant within the elastic limit.*

Experiments show that Hooke's law is approximately true for most engineering materials. For steel it is very nearly true.

For materials like cast iron and concrete, deviation from the law may be considerable. For these materials, however, the range of stress that may be used *in practice* is such that within this range Hooke's law may be used without introducing excessive errors.

Tension.—If a prism (wire or rod) of length L and sectional area A is subjected to an axial pull P, the unit tensile stress is $S_t = P/A$. If λ is the elongation of this prism, the unit strain is $\epsilon = \lambda/L$. Hence designating the ratio of unit stress to unit strain by E_t,

$$E_t = \frac{S_t}{\epsilon} = \text{constant} \tag{a}$$

for all values of P provided the elastic limit is not exceeded. This constant E_t is called the *modulus of elasticity in tension.* It is also known as Young's modulus or the modulus of stretch. The meaning of the word "modulus" in this connection is "measure." Therefore E_t is a measure of the elasticity of a material in tension.*

Compression.—If a prism is subjected to a simple compressive stress, then, within the elastic limit, the ratio of unit compressive stress to unit compressive strain is a constant. That is,

$$E_c = \frac{S_c}{\epsilon} = \text{constant.} \tag{b}$$

This constant E_c is called the *modulus of elasticity in compression.*

Shearing.—In the case of an elemental prism (Fig. 39), the shearing stresses on the four faces of the prism are all of the same intensity (Art. 19). If S_s is the intensity of the shearing

* Popularly, a body is said to possess great elasticity when it is easily deformed and is quick to recover. Rubber, for instance, is commonly thought of as possessing great elasticity. In technical literature, however, elasticity has a very different meaning. Technically, elasticity is the property which causes a body to resist deformation and afterwards to recover its original shape and size. Accordingly, the greater the resistance a body offers to a deformation of a given amount, the greater the elasticity of that body. If a bar of steel and a bar of rubber of the same length and sectional area are stretched the same amount, the force required to stretch the steel bar will be many times that required to stretch the rubber bar. Technically, therefore, the measure (modulus) of elasticity of steel is much greater than that of rubber (about forty thousand times as great). In general, since $E_t = S_t/\epsilon$, it is evident that E_t becomes very large when a large unit stress S_t results in a very small unit strain ϵ.

stresses on the four faces of the elemental prism, and δ is the unit shearing strain (angle of shear measured in radians, Art. 24), then, within the elastic limit, the ratio of unit shearing stress to unit shearing strain is a constant. That is,

$$E_s = \frac{S_s}{\delta} = \text{constant.} \tag{c}$$

The constant E_s is called the *modulus of elasticity* in shear (shear modulus, modulus of rigidity).

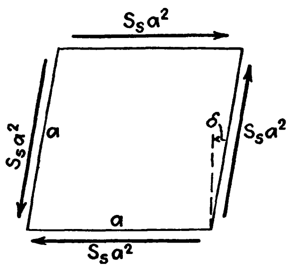


FIG. 39.

There are then three moduli of elasticity for each kind of material, one for tension, one for compression, and one for shearing. For most engineering materials, however, the modulus of elasticity for tension and that for compression are approximately equal so that the same symbol E may be used for both. That is,

$$E = \frac{S}{\epsilon} \tag{d}$$

where S = unit stress (tensile or compressive).
ϵ = unit strain produced.

Now,

$$\text{Modulus of elasticity} = \frac{\text{unit stress}}{\text{unit strain}}.$$

Unit strain, however, is an abstract number (Note, Art. 24). Hence the *modulus of elasticity* is *of the same dimensions as the unit stress.* For example, if unit stress is expressed in pounds per square inch, then the modulus of elasticity is in pounds per square inch.

Example I.—A steel rod of length L = 20 ft. and sectional area A = 1.5 sq. in. is subjected to an axial pull of P = 12 tons. The elongation of the rod is λ = 0.128 in. Find unit stress S, unit strain ϵ, and modulus of elasticity E.

$$S = \frac{P}{A} = \frac{24{,}000}{1.5} = 16{,}000 \text{ lb./sq. in.}$$

$$\epsilon = \frac{\lambda}{L} = \frac{0.128}{20 \times 12} = 0.000533 \text{ (an abstract number).}$$

$$E = \frac{S}{\epsilon} = \frac{16{,}000}{0.000533} = 30{,}000{,}000 \text{ lb./sq. in.} = 15{,}000 \text{ tons/sq. in.}$$

Note.—The elastic limit for structural steel in tension is about 35,000 lb./sq. in. Hence a stress of 16,000 lb./sq. in. is within the elastic limit.

Example II.—If the intensity of the shearing stresses on the four faces of an elemental steel prism is $S_s = 9000$ lb./sq. in. (which is within the elastic limit) and the angle of shear is $\delta = 0.00075$ radian, find the modulus of elasticity in shear.

$$E_s = \frac{S_s}{\delta} = \frac{9000}{0.00075} = 12{,}000{,}000 \text{ lb./sq. in.}$$

27. The ultimate stress or ultimate strength for a given material is the maximum unit stress that can be developed in that material. For instance, given a structural steel rod of area $A = \frac{1}{2}$ sq. in. The maximum pull this rod can sustain is about 30,000 lb. That is, the maximum unit stress that can be developed in this rod is $S = P/A = 60{,}000$ lb./sq. in. Hence 60,000 lb./sq. in. is the ultimate stress (ultimate strength or ultimate limit) for this rod in tension.

For each material there is an ultimate stress for each kind of stress, *i.e.*, one for tension, one for compression, and one for shearing.

28. The variations in the properties of different pieces of the same kind of material may be considerable. For example, in the case of hemlock, the ultimate tensile strength of different pieces may range from about 6500 lb./sq. in. to about 13,500 lb./sq. in. On the other hand, the manufacture of steel has reached such a degree of perfection that steel may be depended upon not to vary much from the prescribed standard for that particular kind of steel.

The values of the various properties of a few of the more commonly used materials are given in Table I (see page 36). Each value is an average of the values found experimentally from a great many pieces of that kind of material. In the last column, the percentage variation in the tensile strength is given for each material. Thus the tensile strength of structural steel is given as 60,000 lb./sq. in. and the percentage variation as 9 per cent (approximately). Hence the ultimate tensile strength of structural steel ranges from about 55,000 lb./sq. in. to about 65,000 lb./sq. in.*

* The 1935 Specifications for Railway Bridges prescribe steel with a tensile stress of 60,000 to 72,000 lb./sq. in.

TABLE I.—ELASTIC LIMIT, ULTIMATE LIMIT, MODULUS OF ELASTICITY
Average values of stress in pounds per square inch

Material	Tension		Compression		Shear		Modulus of elasticity		Range in values, per cent
	Elastic limit	Ultimate stress	Elastic limit	Ultimate stress	Elastic limit	Ultimate stress	Tension and compression	Shear	
Structural steel	35,000	60,000	35,000	*	25,000	50,000	30,000,000	12,000,000	9
High-carbon steel (0.60 % carbon)	60,000	100,000	60,000	......	36,000	80,000	30,000,000	12,000,000	
Structural nickel steel (3.5 per cent nickel)	50,000	90,000	55,000	......	......	......	30,000,000	12,000,000	15
Gray cast iron	6,000?	25,000	......	90,000	......	......	15,000,000	6,000,000	50
Wrought iron	25,000	50,000	25,000	......	......	40,000	27,000,000	10,000,000	15
Wood†	......	......	......	On end of the grain, 2,800 to 4,500	......	Parallel to the grain, 200 to 500	1,000,000 to 2,000,000		

* Ductile materials show no well-defined ultimate stress in compression.

† Range of values for different kinds of wood. Full-size structural timber with ordinary defects. For small straight-grained test specimens, kiln dried, the values may be increased by as much as 100 per cent.

29. Working Stress or Allowable Unit Stress. Factor of Safety.—The working stress or allowable unit stress is the maximum unit stress that is deemed safe to use when designing a member to carry a given load, or when calculating the load a given member may safely carry. As an example, suppose it is required to find the sectional area of a steel rod that is to carry a tensile axial load $P = 9000$ lb. If 18,000 lb./sq. in. is considered to be the maximum unit stress (working stress) that may be used when calculating the area of the section of this rod, then

$$A = \frac{P}{S} = \frac{9000}{18{,}000} = 0.50 \text{ sq. in.}$$

The *factor of safety* is the ratio of the ultimate unit stress to the working stress. That is,

$$\text{Factor of safety} = \frac{\text{ultimate stress}}{\text{working stress}}.$$

If the ultimate stress for a particular piece of steel is 64,000 lb./sq. in. and if a factor of safety of four (4) is to be used, then the working stress is

$$\frac{64{,}000}{4} = 16{,}000 \text{ lb./sq. in.}$$

30. Selection of the Working Stress.—The selection of the working stress or of the factor of safety to be used for a material in a given case depends upon the following considerations:

1. Economy dictates that the working stress should be taken as high in value as is consistent with safety. If the working stress is high, less material will be needed in a given structure. This reduces the cost of structure, which is always important.
2. The working stress should always be taken less than the elastic limit. Even if the stress is less than the ultimate stress but greater than the elastic limit, it is not satisfactory since the structure will be permanently and progressively deformed.
3. Working stresses should be selected low enough to give a sufficient margin of strength to guard against
   a. Uncertainties in the loading. Some loads cannot be determined definitely, especially if there are possible future or accidental increases.
   b. Uncertainties in the stresses. The nature of the structure may be such that the critical internal stresses can be only approximately determined.

c. Uncertainties in the properties of the material. When a working stress is selected for a material, it must be remembered that the properties of a particular piece of this material may be appreciably less than that given in a published table of values.

d. The deterioration of the material due to rust, decay, wear, age, chemical or electrical actions, etc.

4. Seriousness of failure. If the failure of a structure is not likely to result in loss of life or in heavy monetary loss, the unit stresses may be taken higher than for a similar structure if failure endangers life or involves heavy monetary loss.

TABLE II.—WORKING STRESSES FOR STATIC LOADS
Pounds per square inch
Approximate factors of safety in parentheses ( )

Material	Tension	Compression	Shear
Structural steel	16,000 to 20,000 (4)	16,000 to 20,000 (4)	10,000 to 12,000 (5)
Recommended for ordinary use	18,000 (4)	18,000 (4)	12,000 (5)
Gray cast iron	4,000 (6)	16,000 (7)	3,000 (8)

Structural timber	Tension*	Compression		Shear parallel to grain
		Parallel to grain	Normal to grain	
Douglas fir	1,300	1,000	225	90
Western hemlock	1,100	900	225	75
Yellow pine	1,300	1,000	225	125
Soft pines	800	750	150	85
Oak	1,200	900	375	125

* Unless otherwise directed, these stresses should be used also for extreme fiber stresses in flexure (see Art. 173).

The selection of the proper working stresses to be used in a particular case may not be a simple matter. A number of factors enter which can be evaluated only by experts who have had long experience in designing structures and in observing them either under test or in actual service. To protect the public, all important engineering constructions are subject to specifications that embody the consensus of opinion of the best designers. In particular, the materials that may be used and the working

stresses for these materials are specified for the various kinds of construction. To a large extent, *working stresses are standardized.*

31. The working stresses for a given material need not be the same for two different kinds of construction. For instance, the working stresses for a wooden roof truss may be taken higher than those for a wooden railway trestle. The uncertainties in a roof truss are less than those in a railway trestle. Moreover, a roof truss is protected from the weather while a railway trestle is not.

Table II gives the working stresses for a few of the more commonly used engineering materials. In certain cases it may be desirable to use a stress larger or smaller than the tabulated value. In problems in this text, unless otherwise directed, use the tabular values.

PROBLEMS

2. A weight of 70 lb. is suspended from a wire 0.07 in. in diameter. What unit stress is induced in the wire? *Ans.* 18,200 lb./sq. in.

3. In Problem 2, what is the unit elongation if $E = 30{,}000{,}000$ lb./sq. in.? If the wire was originally 400 ft. long, what is the total elongation?
Ans. $\epsilon = 0.000607$; $\lambda = 2.91$ in.

4. A steel plate, $\frac{3}{4}$ in. thick and 7 in. wide, is subjected to a pull of 48,000 lb. Find the largest diameter of hole that may be drilled at the center of the plate if the unit stress is not to exceed 16,000 lb./sq. in.
Ans. 3 in.

5. A concrete pier is to support a load of 960,000 lb. uniformly distributed. The American Institute of Concrete Construction specifies that the allowable stress may be 0.4 of the ultimate. Required the section area of the pier. Take the ultimate for concrete as 3000 lb./sq. in.

Assume the stress in a member as 0.4 of the ultimate. What factor of safety does this imply? *Ans.* 5.55 sq. ft.; 2.5.

6. Brick weighs 120 lb./cu. ft. How high may a circular brick chimney be if owing to its own weight the compressive stress at the base of the chimney is not to exceed 1080 lb./sq. in.? *Ans.* 1296 ft.

7. One of the main cables in the General U. S. Grant Bridge across the Ohio River was designed for a total tension of 2,380,000 lb. The cable is 30 in. in diameter and consists of 1458 parallel wires each 0.162 in. in diameter.

The wires are cold drawn, their average ultimate strength being 230,000 lb./sq. in. What factor of safety was used in the design of the cable?
Ans. 2.94.

8. Figure 40 represents a steel bolt $\frac{5}{8}$ in. in diameter with a square bolt head. The bolt is under a tension of $P = 5000$ lb. See figure for dimensions. Find (*a*) the unit tensile stress in the shank; (*b*) the unit shearing stress in the bolt head, assuming shearing on a cylindrical surface of the

same diameter as the bolt; (c) unit compressive stress under the surface of the bolt head, assuming stress uniformly distributed.

Ans. 16,300 lb./sq. in.; 5090 lb./sq. in.; 11,420 lb./sq. in.

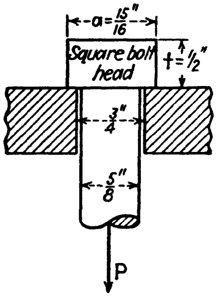


FIG. 40.

9. Find the length of a lead rod that will be on the point of rupturing in tension under its own weight when hung vertically. The weight of lead per cubic foot is 700 lb. Assume the cross-sectional area to be A (or dA). S_t(ult.) = 1800 lb./sq. in.

Ans. 370 ft.

10. A rod $\frac{3}{4}$ in. square is 50 ft. long when unstrained. Using the values for structural steel, find the total stress and also the unit stress when the total elongation of the rod under a tensile force is 0.320 in. State whether or not the elastic limit is exceeded. *Ans.* 9000 lb.; 16,000 lb./sq. in.

11. A steel bar is 2 in. wide and $\frac{1}{2}$ in. thick. Under a total axial pull of 60,000 lb., what are the induced normal and shearing stresses per unit area on sections making angles of 40°; 45°; 50° with the cross-section?

Ans. For 40°, S_n = 35,200 lb./sq. in.; S_s = 29,500 lb./sq. in. For 45°, $S_n = S_s$ = 30,000 lb./sq. in.

12. Figure 41 represents the left end of a wooden member (western hemlock) subjected to a pull P. A rectangular dowel pin holds the member in equilibrium (pin not shown). Assume the pressure between pin and member uniformly distributed. Dimensions shown in figure. Using the values

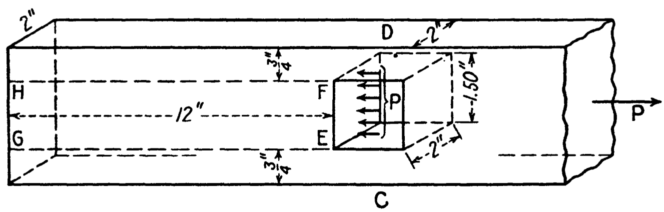


FIG. 41.

given in the table for safe stresses (1100, 900, 75), find the safe value of the pull P. Investigate tensile stress across the section CD; strength in compression against EF; and the combined strength in shear along the two surfaces HF and GE. *Ans.* 3300 lb.; 2700 lb.; 3600 lb.

13. A vertical plate, 3.5 ft. square, covers an opening in a dam. The upper edge of the plate is 4 ft. below the water surface. Construct the stress solid for the plate. Find the total water pressure on the plate.

Ans. 4400 lb.

14. One end of a roof truss takes the form shown in Fig. 28 (Art. 16). The member M is to be square and M is to have the same width as N. P = 4 tons and α = 50°. It is required to find (a) the dimensions of M if the allowable stress in M is 900 lb./sq. in.; (b) the depth c of the notch if the allowable bearing stress along the surface BE is 1000 lb./sq. in.;

(*c*) the length a of the surface BD if the allowable (average) shearing stress is 120 lb./sq. in. *Ans.* 3 in. square; $c = 1.7$ in.; $a = 14.3$ in.

15. A block of hard wood, 6 in. square, supports a uniformly distributed load W. The grain of the wood makes an angle θ with the right section (Fig. 9, Art. 7). The ultimate compressive stress, normal to the grain, is $S_n = 3200$ lb./sq. in. and the ultimate shearing stress, parallel to the grain (along the grain), is $S_s = 800$ lb./sq. in. For what value of θ and for what value of W will the block be as likely to fail in compression as in shear?

Ans. $\theta = \tan^{-1} \frac{1}{4} = 14°$; $W = 122{,}200$ lb.

CHAPTER II

TESTING MATERIALS

LABORATORY TESTS IN TENSION, COMPRESSION, AND SHEAR

32. Tensile Tests. Test Pieces.—A complete and detailed treatment of "testing materials" is far beyond the scope of this text. Special treatises are available, and usually separate technical courses are given covering this subject. In this chapter, only the most fundamental facts are presented, with the hope that the reader may be led to appreciate the importance of the subject and may obtain some idea of the methods used to determine the properties of the materials employed in engineering.

33. Standardized Tests.—To make the results of tests of different pieces comparable with each other, the American

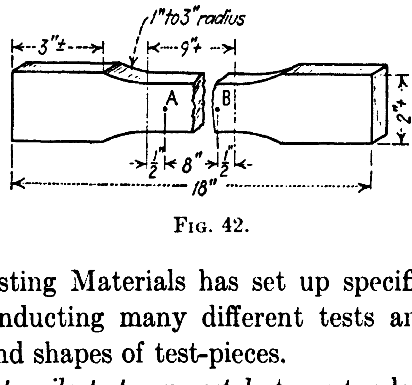


FIG. 42.

Society for Testing Materials has set up specifications for the methods of conducting many different tests and has adopted definite sizes and shapes of test-pieces.

For making *tensile tests on metals* two standard gage-lengths have been adopted, an 8- and a 2-in. gage-length.

Figure 42 shows the dimensions of an 8-in. test-piece, the strain being measured over the 8-in. gage-length $A \ . \ . \ . \ B$.

34. Testing Machines.—In Fig. 43, one type of testing machine is shown. It is called the Riehlé testing machine. The test-piece T is in place ready for a tensile test. The upper end of the test-piece is gripped firmly in the jaws or chuck of the upper or fixed head D. The lower end is gripped similarly in the jaws or chuck of the movable head L. As L is pulled down, the test-piece is subjected to tension. The magnitude of this tension is

measured by the ordinary process of weighing, the total tension being indicated on the beam K with the counterpoise in balance.*

Commercial types of such machines are built ranging in capacity from a few thousand pounds to ten million pounds.

35. The fundamental principles underlying the construction of a machine such as that shown in Fig. 43 are embodied in the more or less ideal machine shown diagrammatically in Fig. 44. That is, in Fig. 44 the essential details are shown and these have been slightly modified for convenience of description.

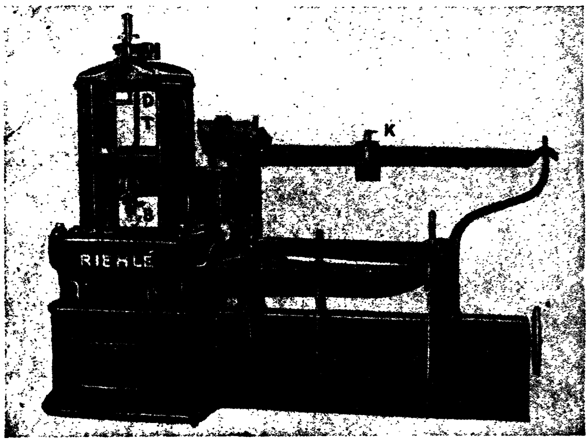


FIG. 43.—Riehlé U.S. standard vertical screw power testing machine, two rotating screw type, 100,000 lb. (50,000 kg.) capacity.

B is a *weighing table* corresponding to the platform of an ordinary platform scale. By means of knife-edges (two of which are represented, E and E') B rests on four levers, two short and two long (only one pair is represented, F and F'). By means of a knife-edge, each lever in turn rests on a heavy fixed base H (only two of these knife-edges are represented, G and G'). The four levers meet at I. The whole arrangement in principle is the same as that of an ordinary commercial weighing scale and any pressure exerted down on B may be measured by the ordinary process of weighing.

The *movable head* L is driven by two (sometimes four) *straining screws* M and M'. These screws pass freely through the weighing table B and then through the base casting H and are held in place at N and N'. That is, the straining screws can turn but cannot move up or down. N and N' are gears firmly fastened to the lower ends of the straining screws. P is a driving gear (method by which P is driven is not shown in figure). As P is made to turn, the straining screws M and M' are made to turn and the movable head L is pulled down. Since the straining screws pass freely through the

* For compression tests, the test-piece is placed between the movable head L and the table B.

weighing table B they do not directly affect the pressure on the table. If a test-piece T is to be tested in tension, it is placed as shown in the figure and the straining screws are turned so that the movable head L pulls down on the test-piece and therefore down on the *fixed head* D. This downward pull on D is transmitted through the columns Q on to the weighing table B and its magnitude is indicated on the beam K when in balance. To obtain a convenient zero reading the adjustable counterweight CW is available.

If a test-piece is to be tested in compression, it is put between the movable head L and the table B. As L is pulled down, the test-piece exerts a pressure directly on B.

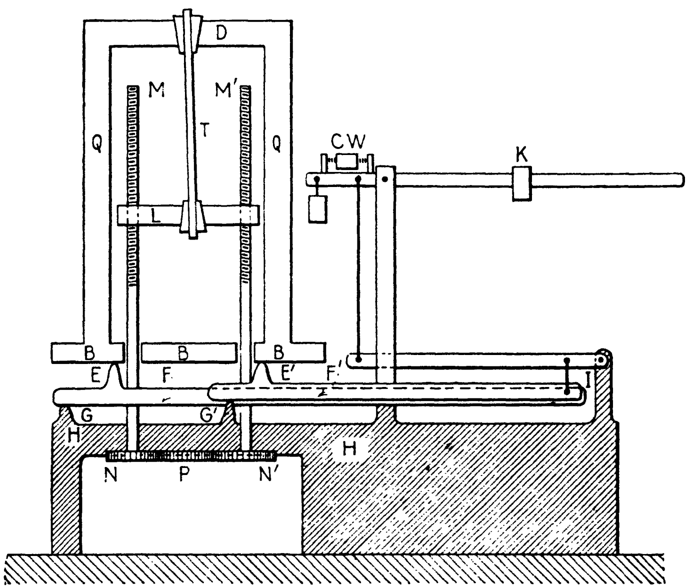


FIG. 44.—Diagrammatic sketch of testing machine.

Note.—In the Olsen machine the gearing at N and N' is such that the straining screws move up or down but do not rotate when the machine is in operation.

36. Extensometers or Strain Gages.—To measure the elongation of the gage-length of a test-piece, an "extensometer" is used, one type being shown in Fig. 45. As the title of the illustration shows, this is known as the Berry strain gage. The dial D measures (to 0.0001 in.) the amount by which the conical points A and B are separated. One point A is fixed in the frame F, and the other point B is movable. The arrangement is such that by means of levers and gears any small motion of B produces an appreciable motion of the dial hand.

Below the figure of the strain gage is shown a clamp, and at the bottom is illustrated an accurate punch used to mark the gage points on the test-piece to make an 8-in. gage-length.

The extensometers usually employed measure elongations to an accuracy of 0.001 or 0.0001 in. Instruments are made, however, which will measure to a precision of 0.00001 in. or to an even higher degree of accuracy.

37. Ductility. Brittleness.—*Ductility* is that property by virtue of which a material may be drawn out by tension resulting in a permanent increase in length (set) and in a permanent decrease in sectional area. This drawing-out takes place after

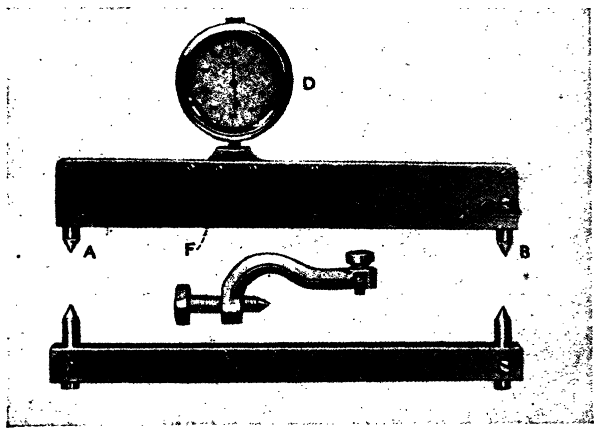


FIG. 45.—Eight-inch instrument.

the elastic limit is passed. A perfectly ductile material could be drawn out indefinitely into a fine wire without breaking.

Most structural materials possess the property of ductility to some extent. It is customary, however, to designate those materials as *ductile* which may be drawn out considerably before rupture takes place, and to designate those materials as *brittle* which can be deformed permanently only to a very limited extent. Accordingly, soft steel is ductile, while hard steel is brittle. If a rod of soft steel, 8 in. long, is subjected to tension beyond its elastic limit, the permanent elongation will exceed 30 per cent of the original length before it breaks. On the other hand, a rod of hard steel will break suddenly, the permanent elongation being relatively very small. The distinction between

a ductile steel and a brittle steel is more clearly shown in the next article.

38. Stress-strain Diagrams.—When a material is tested in tension for its mechanical properties, it is common practice to make a graphic report of the observed quantities obtained in the test. Such a graph or curve, termed the *stress-strain diagram,* presents the facts of the test in the most striking form. It shows the degree of accuracy attained in the testing, and from it may be found the important values of modulus of elasticity, proportional limit, ultimate limit, and breaking unit stress. An indication of the toughness and the brittleness of the material is also given.

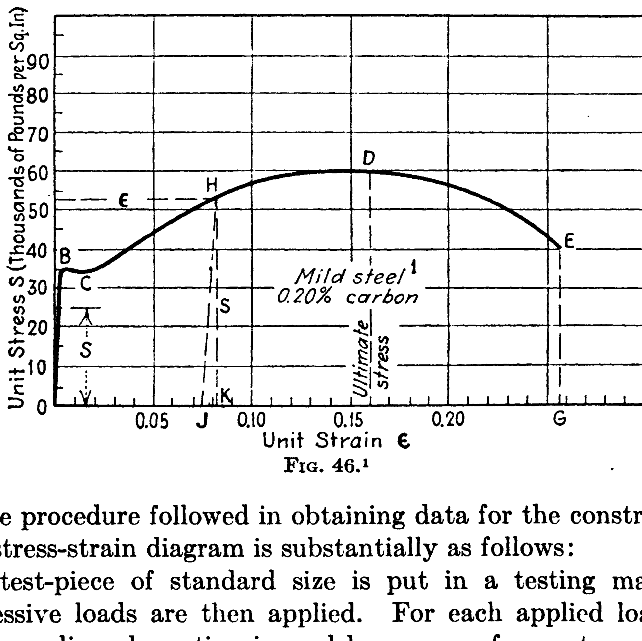


FIG. 46.[1]

The procedure followed in obtaining data for the construction of a stress-strain diagram is substantially as follows:

A test-piece of standard size is put in a testing machine. Successive loads are then applied. For each applied load the corresponding elongation is read by means of an extensometer. If λ is the elongation for a particular load P, then for this load the unit stress is $S = P/A$ and the unit strain is $\epsilon = \lambda/L$. With S as ordinate and ϵ as abscissa, the point represented by S, ϵ, is plotted (Fig. 46 or Fig. 47). If this is done for successive values of P ranging from zero to that value for which the test-piece

[1] Steels may be divided into three kinds, *soft, medium,* and *hard.* Wide variations exist within each kind. Structural steel (0.20 per cent carbon) is a soft steel usually designated as *mild* steel.

breaks, a series of points is determined. A line drawn through these points gives the stress-strain curve. Such a curve, therefore, shows the relation between the unit stress and the unit strain at each stage during the elongation of the test-piece.

The stress-strain diagram for soft steel in tension is shown in Fig. 46, and that for hard steel is shown in Fig. 47. The two curves are more or less typical. By this is meant that the stress-strain diagram for any ductile material is likely to be quite similar to the diagram for soft steel in tension, and that the diagram for a brittle material is more or less similar to that for hard steel in tension. A discussion of the two stress-strain diagrams shown in Figs. 46 and 47 will cover the important characteristics of curves for many materials. In Art. 44, the stress-strain diagrams for several of the commonly used engineering materials are shown.

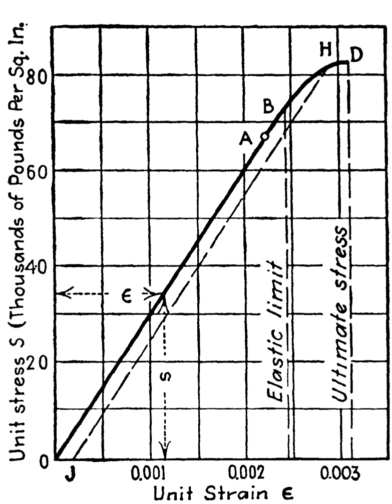


FIG. 47.—Hard steel in tension (brittle steel.)

39. Elastic Limit. Proportional Limit. Modulus of Elasticity.—The *elastic limit* corresponds to the point B (Fig. 46 or Fig. 47). So long as the experiment does not go beyond the stage represented by the point B, the test-piece will recover its original length when the pull is removed. If, however, the experiment reaches a stage represented by a point beyond B, say the point H, and if then the pull is removed the test-piece will not fully recover. There will remain a permanent strain (set) represented by the line OJ.

Note.—For convenience, the point B in the diagram may be called the elastic limit. It should be understood, however, that the *elastic limit* is the unit stress represented by the ordinate to the point B. A similar statement will apply to the other points on the curve.

Proportional Limit.—The stress-strain diagram given in Fig. 46 or Fig. 47 is very nearly a straight line up to the point A which is a little below the elastic limit B. Accordingly, up to

the point A the unit stress is very nearly proportional to the unit strain. The unit stress represented by the ordinate to the point A is called the proportional limit. That is, the *proportional limit* is the maximum unit stress within which the ratio of unit stress to unit strain may be considered constant. For convenience, the point A in the curve may be called the proportional limit.

The *modulus* of *elasticity* was defined as the ratio of unit stress to unit strain (Art. 26). Hence, within the proportional limit,

$$E = \frac{S}{\epsilon} = \tan \angle JOA = \text{constant (nearly)}.$$

40. The values obtained for the *proportional limit* and the *elastic limit* of a material are approximations. Among other things, these values depend upon the care with which the test is made and upon the sensitiveness of the instruments used. If very sensitive instruments are used, a permanent set may be obtained where, with the instruments ordinarily used, no set is observed. Moreover, if tests are carefully made and if the stress-strain diagram is plotted to a large scale, the lower part of the diagram, *i.e.*, the part OA, will not be a straight line. In most cases, however, it will be very nearly straight. Hence, to make the results of different tests on a material comparable with each other and to allow for slight variations from the straight-line relation, certain definite methods of procedure are recommended for determining the proportional limit, *i.e.*, for determining that point on the stress-strain diagram up to which, for *practical purposes*, the diagram may be considered a straight line. For details concerning methods of procedure the reader is referred to textbooks on the testing of materials.

For steel and some of the other more important engineering materials, the proportional limit and the elastic limit, as commonly found, have approximately the same numerical value. This is particularly true of ductile materials. Moreover, the more accurately the tests are made, the more nearly do the two limits agree. Hence no distinction is commonly made between the proportional limit and the elastic limit. *Since the proportional limit is more easily obtained, its value is commonly reported as the elastic limit.* The proportional limit is frequently called the *proportional elastic limit.*

Note.—Unless a statement to the contrary is made, the "elastic limit" should be interpreted to mean the "proportional limit."*

41. Yield-point. Commercial Elastic Limit.—When a prism of *wrought iron* or *soft steel* is stretched a little beyond the elastic limit B, the bar suddenly elongates with little or no increase in the load. This stage of the test is represented by the point C in the stress-strain diagram for soft steel (Fig. 46). The unit stress corresponding to the point C is called the yield-point; *i.e.*, the *yield-point* is the unit stress at which the bar begins to elongate with little or no increase in the load. When the yield-point is reached, the prism acts as though somewhat viscous.

If a piece is tested in a testing machine (Fig. 43), the beam K suddenly drops when the yield-point is reached. Hence the yield-point is readily determined. In commercial testing, it is customary to determine the yield-point instead of the elastic limit in the case of wrought iron and soft steel. The yield-point is sometimes called the *commercial elastic limit.*

Note 1.—A brittle material (hard steel, cast iron, concrete, etc.) has no sharply defined yield-point.

Note 2.—Some ductile materials (copper for instance) show no well-defined yield-point.

Note 3.—The division line between ductile and brittle materials is not clearly marked. For instance, if steels are arranged in order of their ductility, it will be found that there is a gradual transition from decidedly ductile to decidedly brittle steels. The statements made in this article as well as some of those made in the previous articles in this chapter must be modified when applied to materials that are near the division line.

42. Ultimate Strength, Ultimate Stress, or Ultimate Limit.—The point D in the stress-strain diagram (Fig. 46 or Fig. 47) gives the greatest unit stress that can be developed in the test-piece. The unit stress corresponding to the point D is called the *ultimate strength* or the *ultimate stress* or the *ultimate limit,* and the corresponding total load (pull) is the *ultimate load.*

* To determine the true elastic limit, the test-piece is subjected to a gradually increasing load and, after each definite increment of the load, the load is gradually removed. This is continued until a permanent set appears. The true elastic limit cannot be determined from the stress-strain diagram. The proportional limit, however, is determined from the stress-strain diagram. Accordingly, it is much easier to determine the proportional limit.

When the point D is reached, a brittle material acts very differently from a ductile material. A brittle material suddenly breaks when its ultimate stress is reached as is indicated in Fig. 47. A ductile material, however, begins to "neck" when its ultimate stress is reached, *i.e.*, it is drawn out at some locality and the sectional area of the piece is greatly reduced before it breaks. Figure 48*a* represents a test-piece of soft steel just before necking began, and Fig. 48*b* shows this piece after necking took place at N. The reduction in sectional area at N may be as large as 50 per cent before the piece breaks. As soon as necking begins, the elongation may be continued under a decreasing load as is indicated in Fig. 46.

43. Compression tests usually are made on short blocks, *i.e.*, on cubes or short prisms of the material. Experiments show that the elastic properties of *some* materials in compression may differ markedly from the properties in tension.

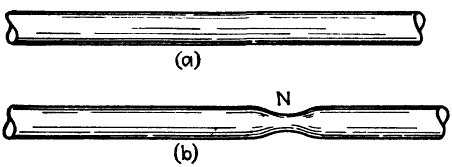


FIG. 48.

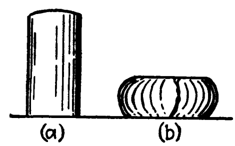


FIG. 49.

In the case of a ductile material in compression, as soon as the yield-point is exceeded the material begins to *flow laterally; i.e.*, it will be permanently flattened. Figure 49*a* shows a short prism of very soft steel before it was subjected to compression, and Fig. 49*b* shows the same piece after it had been compressed beyond its yield-point. The material has noticeably flowed laterally. The extent to which a piece of ductile material may be deformed in this way cannot be very definitely predicted, and for this reason a ductile material may show no well-defined ultimate compressive strength. A brittle material under compression shows no visible deformation and fails suddenly.

Compression tests are made principally on brittle materials such as cast iron, concrete, brick, and stone—materials which are not suitable for tension members—and on wood.

The properties of a hard steel in compression are very similar to those for this steel in tension. The same is true for soft steel up to the yield-point. Hence commercial tests on steel need be made only for the tensile values.

44. Stress-strain diagrams may be constructed for the various materials commonly used in engineering construction. Figure 50 gives the diagrams for wood in tension and in compression, for cast iron (tension and compression), and for three grades of steel (low carbon, medium carbon, and high carbon, *i.e.*, soft steel, medium steel, and hard steel, all in tension). The broken lines indicate that the curves extend beyond the horizontal range of the figure. Figure 51 gives the complete diagrams for the three steels on a greatly reduced horizontal scale.

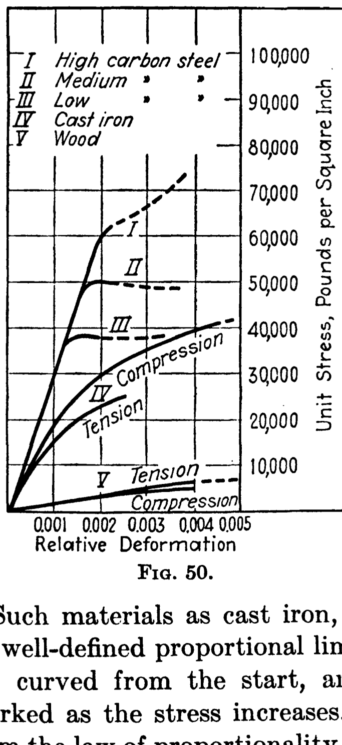


FIG. 50.

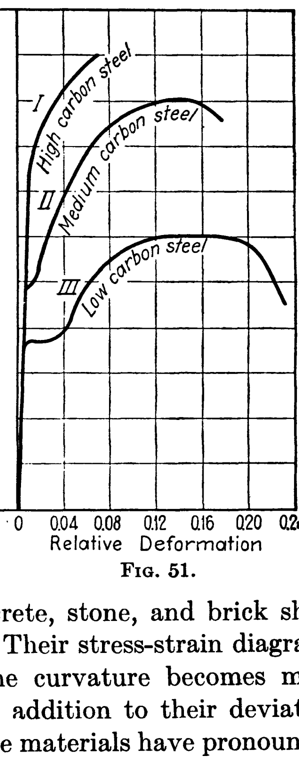


FIG. 51.

Such materials as cast iron, concrete, stone, and brick show no well-defined proportional limit. Their stress-strain diagrams are curved from the start, and the curvature becomes more marked as the stress increases. In addition to their deviation from the law of proportionality, these materials have pronounced and erratic variations in their properties. Accordingly, the allowable stress for one of these materials is taken considerably below the ultimate stress, bringing the allowable stress down to a value below which the stress-strain diagram is only slightly curved. Hence the modulus of elasticity for such a material as cast iron or concrete may be taken as

$$E = \frac{S}{\epsilon}$$

where S = allowable stress.

ϵ = corresponding unit strain.

Illustration.—The allowable stress for cast iron in compression is about 16,000 lb./sq. in. Up to this point the diagram is only slightly curved (Fig. 50). Therefore, no great error will be made by assuming that up to the allowable stress the ratio of unit stress to unit strain is constant.

45. Actual and Nominal Unit Stress.—When a test-piece is subjected to tension its sectional area decreases, and when it is

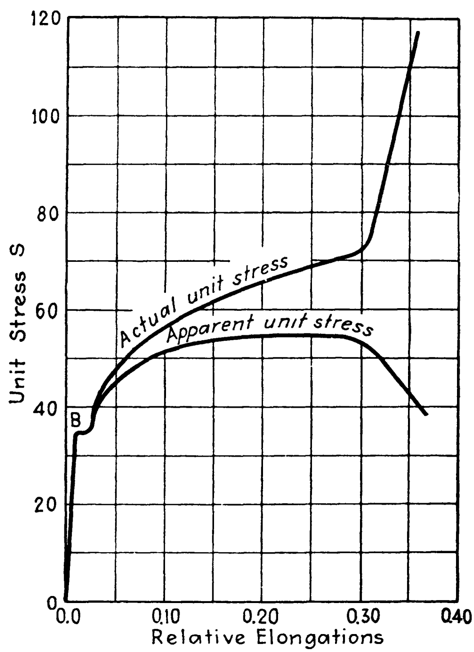


FIG. 52.—Actual and apparent unit stress.

subjected to compression its sectional area increases. To obtain the *actual unit stress* in the test-piece at any special stage during the operation of testing the piece, *for strict accuracy* the load should be divided by the actual sectional area of the test-piece at that particular stage of the test. It is customary, however, to express the unit stress at any particular stage as the unit stress based on the original sectional area, *i.e., to divide the total load by the sectional area of the test-piece before the test began.* The unit stress thus obtained is evidently not exactly correct,

but it is a simple and standard practice. This stress is termed the *nominal* or *apparent unit stress.*

For brittle materials, and for ductile materials up to the elastic limit, the decrease (or increase) in the sectional area of the test-piece is relatively so small that no appreciable error is made if the sectional area is taken as that before the test began. The actual and the nominal stress-strain diagrams are practically identical, therefore, for a brittle material and up to the elastic limit for a ductile material. Beyond the elastic limit B, the two diagrams for a ductile material differ a great deal, as is shown in Fig. 52.

46. Shear Tests.—There are two kinds of tests for the determination of the shear properties of materials, *viz., torsion tests*

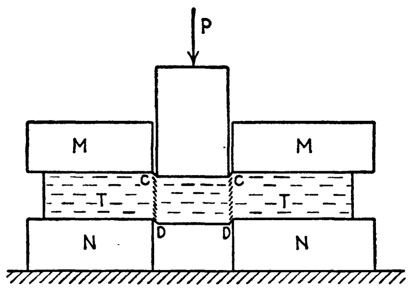


FIG. 53.

and tranverse shear tests. Torsion tests will be considered later (see Chap. V). Transverse shear tests are made primarily to determine the ultimate strength for a material in shear.

There are various types of machines used for transverse shear tests. The fundamental principle underlying them is embodied in the simple diagram of Fig. 53. The test-piece T is assumed for convenience as having a rectangular section and is shown as partly sheared along the two sections CD and CD. The ends of T are firmly held between two plates M and N. Note that the test-piece is sheared along two sections CD and CD. Hence, if A is the area of the section of the test-piece, the shearing area is $2A$. If P is the maximum force necessary to shear the piece, the ultimate intensity of the shearing stress (ultimate shearing strength) is

$$S_s = \frac{P}{2A}.$$

Evidently, the value of S_s thus found is the average intensity of the maximum shearing stress on the section. It is not known just how this stress is distributed over the section.

47. Other Tests.—Materials are tested for various purposes. In addition to tests in tension, compression, and shear, it may be necessary to test a material to ascertain its ability to resist corrosion, or to ascertain its plasticity, flexibility, hardness, malleability, etc. (see Art. 51 for fatigue strength).

In Mechanics of Materials, as in other branches of science, theory and experiments go hand in hand. The results of experiments often are the basis of a theory. The theory in turn must be tested experimentally. No theory can be accepted unless it agrees reasonably well with the results of experiments. In later articles, formulas will be developed for the design of beams. These formulas rest on certain facts experimentally determined. It becomes necessary, therefore, to test beams not only for the purpose of ascertaining the properties of a material when used in a beam but also for the purpose of determining the limitations that must be placed upon the formulas that are used to design beams. In like manner, column tests, torsion tests, in fact, tests of various kinds, must be made. Such tests, as a rule, are made on models, members, or structures especially designed for experimental purposes. Sometimes tests are made on full-size members or on structures such as are actually found in practice.

48. Importance of Ductility.—Structural steel is the steel commonly used in buildings, bridges, cranes, etc. It is a soft steel (mild steel) having an ultimate strength of about 60,000 lb./sq. in. Since a hard steel may have an ultimate strength much higher than this, it may be asked why one of the harder steels is not used.

As a rule, if the strength of a steel is high, its ductility is low. Now structural steel is as strong a steel as can be manufactured *at low cost* consistent with the degree of ductility that is deemed necessary for the steel to have. It may be of interest, therefore, to give a few of the advantages of ductility.

1. It frequently happens that there is a concentration of stress at some point in a member. To illustrate, Fig. 54 represents a symmetrical member subjected to two equal but oppositely directed uniformly distributed forces, one at each end. The member contains a hole (a rivet hole, perhaps). Careful

experiments show that the stress may be considered uniformly distributed on any section that is some distance away from the hole. On a section through the hole, however, the stress is not uniformly distributed as is indicated in the figure, the intensity of stress at C and D being much greater than the average intensity of stress on the net section. That is, there is a concentration of stress at C and at D.

The stress on a section is called a *localized stress* when its distribution is influenced by some local condition such as a hole, groove, notch, flaw, fissure, or an abrupt change in section. Localized stresses are found also near the point of application of a concentrated load.

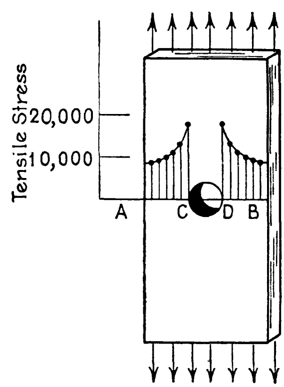


FIG. 54.

If in a ductile material (such as structural steel) a localized stress at some point should exceed its elastic limit, the material at this point will yield; *i.e.*, it will be drawn out without any appreciable increase in the stress at this point. This tends to equalize the stress on that section in that it compels the fibers at the other points in the section to sustain a greater share of the load without necessarily doing any appreciable damage to the member. For instance (Fig. 54), if the intensity of stress at C should exceed its elastic limit, the fibers at this point will yield slightly, and this will compel the other fibers along AC to sustain a greater share of the load. If in a brittle material (such as hard steel) the stress at some point should exceed its elastic limit, the material at this point is apt to fail. A local failure, however, must be regarded as the initial failure of the member as a whole. Ductility, therefore, is of great importance in that it tends to provide insurance against the failure of a member due to localized stresses. Except in extreme cases, *localized stresses in ductile materials are of little significance* (see Fatigue of Metals, Art. 51, for an exception).

2. Frequently, some adjustment must take place among the members or parts of a member before each member or each part will sustain its share of the load. Ductility is very important in that it helps to make such adjustments possible.

If one rivet in a riveted joint carries more than its share of the load, this rivet, if made of ductile steel, will yield slightly when overstressed, thus compelling the other rivets to carry a greater share of the load. If the rivets were made of hard steel the overstressed rivet would be apt to fail, thus compelling the other rivets to carry the whole load.

3. A structure made of ductile steel often is deformed noticeably before it finally collapses. If made of brittle steel, the structure, when overstressed, will fail suddenly without giving special warning by any obvious deformation. Hence, in many cases, ductility is of importance in that the deformation of a structure indicates that the structure is overloaded.

4. It will be seen later (Art. 69) that a ductile steel can absorb the energy due to shocks, blows, and suddenly applied loads better than hard steel. This is of importance in such structures as bridges and cranes.

Note.—When ductility becomes of less importance, a harder steel may be used. The cables of the George Washington Bridge over the Hudson River are made of strands of cold-drawn steel wire whose ultimate strength is about 220,000 lb./sq. in.

49. Measure of Ductility.—In commercial testing, ductility is measured, as a rule, in one of two ways.

1. *Percentage of Elongation.*—A test-piece of a ductile material (such as soft steel) is elongated in a testing machine until it breaks. The two parts of the test-piece are then removed from the machine and are carefully fitted together so that they are relatively in the same position as they were before rupture took place. The elongation of the test-piece (*i.e.*, of the gage-length) is then measured. If λ is the elongation and L is the original length of the gage-length, the percentage of elongation is

$$\text{Percentage of elongation} = \frac{\lambda}{L} \times 100.$$

Now the elongation of the gage-length consists partly of a general elongation, which is practically uniform for the entire length, and partly of a local elongation due to the necking (Fig. 55). The part EF containing the neck will elongate much more than any other part originally of the same length. In the case of mild steel, EF may elongate 100 per cent (double its length), while the elongation of OI (originally 8 in. long) may be only about 30 per cent. Hence, if the percentage of elongation

is to have a definite meaning, the original gage-length must be either specified or understood. Standardized tests, therefore, make the results of different tests comparable with each other.

2. *Percentage of Reduction of Area.*—If A is the original sectional area of the test-piece, and A' the final or minimum sectional area at the neck, then

$$\text{Percentage of reduction of area} = \frac{A - A'}{A} \times 100.$$

In the case of soft steel, the percentage of reduction of area may be as great as 50 per cent; *i.e.*, the sectional area at the neck may be reduced by one-half before rupture takes place.

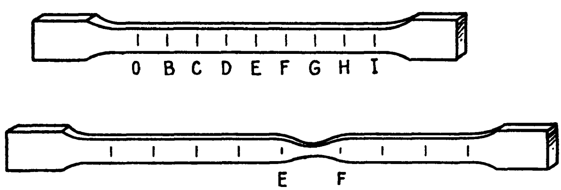


FIG. 55.

50. The physical properties of a manufactured material such as steel may be modified in several ways.

1. *Chemical Composition.*—The properties of steel may be materially modified by the addition of a small amount of one or more of the following elements: carbon, nickel, chromium, silicon, manganese, etc. Each element has a particular influence upon the properties of steel, and by a proper combination of these elements steels of various kinds are made. A steel that contains carbon *only* is called a *carbon steel.* If it contains one or more of the other elements, it is called an *alloy steel.* "Stainless steel," for instance, is an alloy steel usually made by the addition of chromium. *Structural steel* is a carbon steel, the carbon content ranging from about 0.15 to about 0.30 per cent. It can be manufactured on a large scale and at a low cost.

Nickel steel (3 to 4 per cent), sometimes called *structural nickel steel*, possesses properties that are very desirable for high-grade structural work and for machines. It is a ductile steel with an ultimate strength of about 100,000 lb./sq. in. It is, however, too expensive for general use.

To increase the hardness of steel, carbon has long been used. To illustrate how the addition of carbon affects the elastic limit and the ultimate strength of steel, the following table is given.

Amount of carbon, per cent	Carbon steel	Hot rolled
	Elastic limit, lb./sq. in.	Ultimate strength, lb./sq. in.
0.20	35,000	60,000
0.60	60,000	100,000
1.00	80,000	135,000

Note.—A hot-rolled metal is a metal that has been rolled into shapes (rods, bars, beams, channels, etc.) while red hot. A cold-rolled metal is one that has been rolled into shapes while cold.

2. *Overstrain.*—When a material is stressed beyond its elastic limit, it is said to be *overstressed.* When the strain is under consideration, it is said to be *overstrained.*

Overstraining a ductile material may change its physical properties markedly. If a test-piece of soft steel is overstrained, say up to the point H in Fig. 46, and if then the stress is removed, there will remain a permanent set represented by OJ. If later the stress is again applied, it will be found that the stress-strain diagram now is given by JHE. This means that the new elastic limit is very nearly that represented by the ordinate to the point H. That is, the elastic limit has been raised.

Hammering, cold-pressing, cold-drawing, cold-rolling, etc., are all cases of overstrain. The following table compares the elastic limits and the ultimate strengths of two pieces of structural steel, one hot-rolled, the other cold-rolled.

STRUCTURAL STEEL
0.20 per cent carbon

	Elastic limit, lb./sq. in.	Ultimate limit, lb./sq. in.
Hot rolled	35,000	60,000
Cold rolled	60,000	80,000

3. *Heat treatment* consists in heating a material (metal, glass, etc.) to a high temperature and then cooling it either slowly or suddenly. Different materials respond differently to heat treatment. Since steel is the metal of primary interest in this text, it will be advantageous to consider here the heat treatment of steel.

Annealing steel, *i.e.*, heating it to a red heat (800°C. more or less) and then allowing it to cool *slowly*, tends to soften the steel and to make it more ductile. An overstrained steel if properly annealed will recover its original properties. Annealing also makes a steel more uniform internally. Hence, to produce a high-grade soft steel, the steel may be first hammered, rolled, drawn, etc., into shape and then annealed.

Note.—Glass and aluminum are similarly annealed by slow cooling from a high temperature. On the other hand, copper, brass, and bronze are annealed by quenching or quick cooling from a heated state. Slow cooling hardens them.

Hardening. Tempering.—Heating a steel to a red heat and then cooling it by plunging it into water or oil tends to harden it. Hardening raises the elastic limit and the ultimate strength of steel. A hardened steel is apt to be too brittle. When this is the case, the hardened steel is *tempered.* This process consists in heating it to 450°C. (more or less, depending upon the amount of tempering desired) and then allowing it to cool. This reduces the hardness and makes the steel more uniform in texture.

The following table shows how the elastic limit and the ultimate limit of spring steel may be modified by heat treatment:

SPRING STEEL
0.93 per cent carbon

Heat treatment	Elastic limit, lb./sq. in.	Ultimate limit, lb./sq. in.
Annealed	28,000	84,000
Hardened	107,000	188,000
Tempered	85,000	97,000

Note.—Since the properties of a manufactured material such as steel depend both upon the mechanical processes used in its manufacture and also upon its heat treatment, it is to be expected

that two pieces of metal of the same chemical composition may differ in their physical properties.

51. Fatigue Failure. Endurance Limit.—The ultimate unit stress as defined in Art. 27 is often called the *static ultimate stress* since it is the maximum unit stress that can be developed in a material under static conditions (load slowly and gradually applied). The working stress as defined in Art. 29 is based upon the static ultimate stress. Experience shows that this is always permissible if the stress in a member is constant or is repeated only a relatively small number of times.

If a load is applied to a member *many times*, it is called a *repeated load*. The unit stress induced in a member by a repeated load is called a *repeated stress*. For instance, a material may be subjected to a stress that varies between S_1 and S_2, (from S_1 to S_2, S_2 to S_1, etc.). If S_1 and S_2 are of opposite signs (one tension, the other compression), the stress is called a *reversed stress*.

Experiments show that a material may fail if repeatedly subjected to a stress that is less than the static ultimate stress. If there is a reversal of stress, the material may fail even if the stress is less than the static elastic limit for that material. Failure due to the repetition of stress is called *fatigue failure*.

Parts of a machine or structure may be subjected to millions of stress repetitions. It is estimated that, during the lifetime of a steam engine, the piston rod may have a billion reversals of stress. Axles, crankshafts, automobile springs, etc., often fail owing to fatigue. Moreover, no warning of impending failure is given. The member fails suddenly. A knowledge of fatigue failure is therefore of practical importance.

An extensive treatment of fatigue failure is beyond the scope of this text.* The following statements, however, may be considered safe working hypotheses for cases of a stress repeated *many* times:

1. For a given lower stress S_1, there is an upper limit to the stress S_2 (called the *endurance limit*) such that failure will not occur if a stress varying between S_1 and S_2 is applied an indefinite number of times. When a member to be subjected to many repetitions of stress is designed, it is considered good practice to take, as the working stress, not more than one-third of the endurance limit. For instance, for a complete reversal of stress

* For a detailed treatment of the subject, see "Civil Engineering Handbook," McGraw-Hill Book Company, Inc., 1934.

the endurance limit is about one-third of the static ultimate. Accordingly, the working stress should be about one-ninth of the static ultimate (or even less, depending upon the kind of structure under consideration). If the static ultimate of a steel is 60,000 lb./sq. in., then the working stress for that steel subjected to many complete reversals of stress should be between 6000 and 7000 lb./sq. in.

2. If the lower stress is zero, the endurance limit for steel is about one-half of the static ultimate so that the working stress should be taken as about one-sixth of the static ultimate ($\frac{1}{3} \times \frac{1}{2}$), provided the stress is repeated *many* times (millions of times).

3. Provided there is no reversal of stress, the number of repetitions of stress in a member of a structure such as a bridge is not great enough during the lifetime of that structure to make it necessary to consider fatigue failure. Hence, in the design of a bridge, for instance, fatigue failure need not be considered as a rule if there is no reversal of stress. If there is a reversal of stress in a member of a bridge, the allowable stress is lowered.

PROBLEMS

16. A round rod of structural steel, $\frac{5}{8}$ in. in diameter, is 12 ft. long. Using the stress-strain curve of Fig. 46, estimate the length of this rod if subjected to a pull of $P = 15{,}400$ lb. *Ans.* 12.8 ft.

17. Using the stress-strain diagram for high-carbon steel (Fig. 50), find the modulus of elasticity of the steel. *Ans.* $E = 30{,}000{,}000$ lb./sq. in.

18. A flat round-ended punch, $\frac{7}{8}$ in. in diameter, is used to punch a hole through a steel plate. Plate is $\frac{3}{8}$ in. thick. Find the ultimate unit shearing stress if the maximum force applied to the punch is $P = 52{,}000$ lb.

Ans. 50,500 lb./sq. in.

19. A steel rod, 1.25 in. in diameter, is subjected to a gradually increasing pull. Just before failure the pull in the rod was $P = 90{,}000$ lb. and the diameter of the neck was 1.07. Find the maximum nominal unit stress in the rod. Also the maximum actual unit stress.

Ans. 73,400 lb./sq. in.; 100,000 lb./sq. in.

20. In Problem 19, compute the percentage of reduction of area.

21. A bar of ductile steel, $\frac{1}{2}$ in. in diameter, was used in a testing machine. The beam K dropped when the total load was 7500 lb. Compute the yield-point. *Ans.* 38,200 lb./sq. in.

22. A Berry strain gage was applied to an 8-in. gage-length of a steel rod whose stress-strain diagram is given in Fig. 47. What was the reading of the dial of the gage at the elastic limit? Each unit on the dial registers 0.001 in.

Ans. 19.5.

CHAPTER III

TENSION AND COMPRESSION

There are a few formulas pertaining to tension (or to compression) that are of considerable practical importance. These will now be derived for the case of a prism in simple tension. There are also a few propositions that are more or less general in nature but that will be considered here in connection with a prism in simple tension.

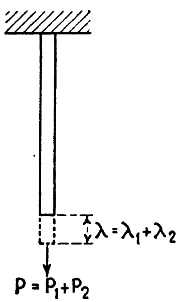


FIG. 56.

52. Law of Proportionality. Law of Superposition.—If λ is the elongation of a (steel) prism due to an axial force P (Fig. 56), then within the elastic limit λ is proportional to P (very nearly). That is, within the elastic limit

$$\lambda = mP \tag{a}$$

where m = a constant (very nearly)

Assume that $P = P_1 + P_2$. The elongation due to P is then

$$\lambda = m(P_1 + P_2) = mP_1 + mP_2 = \lambda_1 + \lambda_2$$

where $\lambda_1 = mP_1$ = elongation due to P_1 acting alone.
$\lambda_2 = mP_2$ = elongation due to P_2 acting alone.

Hence within the elastic limit the elongation due to P_1 and P_2 acting simultaneously may be obtained by adding the elongations due to P_1 and P_2 each in turn acting alone. Or

$$\lambda = \lambda_1 + \lambda_2.$$

Consider now an ideal case and assume that

$$\lambda = mP^n \tag{b}$$

where m and n = constants. Let $P = P_1 + P_2$. The elongation due to P is

$$\lambda = m(P_1 + P_2)^n;$$

that due to P_1 acting alone is

$$\lambda_1 = mP_1^n;$$

and that due to P_2 acting alone is

$$\lambda_2 = mP_2^n.$$

Let us see under what condition $\lambda = \lambda_1 + \lambda_2$. That is, under what condition will the elongation produced by P_1 and P_2 acting simultaneously be equal to the sum of the elongations produced by P_1 and P_2 each in turn acting alone? If $\lambda = \lambda_1 + \lambda_2$, then

$$m(P_1 + P_2)^n = mP_1^n + mP_2^n. \qquad (c)$$

Evidently, the relation expressed by Eq. (c) is possible only if $n = 1$. If $n = 1$, Eq. (b) becomes

$$\lambda = mP.$$

Hence $\lambda = \lambda_1 + \lambda_2$ if (and only if) the elongation is proportional to the force producing it—only if the *law of proportionality* holds.

Considerations of the nature of the foregoing *together with the results of experience* lead to a very important generalization known as the *law of superposition.* This law may be stated as follows:

So long as the law of proportionality is satisfied, the effect of a system of forces acting on a body may be found by combining the effects produced by the individual forces, each force in turn acting alone. That is, *so long as the law of proportionality is satisfied, the effect of one force may be superposed upon the effect of another force to obtain the combined effect produced by both forces acting simultaneously.*

In theoretical mechanics, bodies are considered rigid. Experience warrants the assumption that for rigid bodies the law of proportionality is strictly true. In mechanics of materials bodies are considered deformable, and when the deformation of a body must be considered the law is not always true. In the case of columns, for instance, it cannot be used. Generally, however, the law is nearly true within certain limits, and within these limits the law of superposition is extensively used.

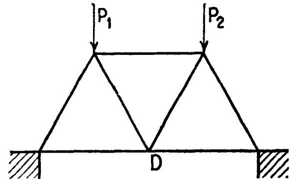


FIG. 57.

Illustration. Fig. 57.—Within the elastic limit, the deflection of the point D produced by P_1 acting alone is very nearly proportional to P_1, and that due to P_2 acting alone is very nearly proportional to P_2. Hence, by using the law of superposition, the deflection of D due to both loads acting simultaneously

may be found by adding the deflections due to P_1 and P_2 each in turn acting alone, provided the elastic limit is not exceeded. If the elastic limit is exceeded in any member of that structure the elongation of that member is no longer proportional to the stress in that member and therefore the law of superposition should not be used. The law of superposition has a wide application in Mechanics and its limitations should be clearly understood.

Note.—The law of proportionality leads to the "method of unit load." For instance, if u is the stress in a given member of a bridge due to a unit load acting at some definite point of that bridge, then the stress in that member due to a load P acting at that point is

$$S = Pu.$$

53. Total Elongation of a Prism under Simple Tension.—Given a prism of length L and sectional area A (Fig. 58). Let λ equal the elongation of the prism when it is subjected to an axial pull P. Since unit stress is $S = P/A$ and unit strain (elongation per unit length) is $\epsilon = \lambda/L$, the modulus of elasticity is

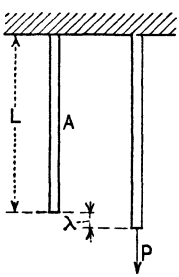


FIG. 58.

$$E = \frac{S}{\epsilon} = \frac{\frac{P}{A}}{\frac{\lambda}{L}} = \frac{PL}{A\lambda}.$$

Or, solving for λ,

$$\lambda = \frac{PL}{AE}. \tag{1}$$

If P/A is replaced by S,

$$\lambda = \frac{SL}{E}. \tag{2}$$

Formula (1) is important. It is useful when solving problems involving the elongation (or contraction) of a prism under simple tension (or compression) within the elastic limit. If the elastic limit is exceeded, E, the ratio of stress to strain, is no longer constant and therefore Eq. (1) is no longer applicable.

Note.—It is necessary to be consistent in the use of units of force and distance. If P is given in pounds and A in square inches, L must be expressed in inches and E in pounds per square

inch. "Mixing units" is one of the most common mistakes made by a beginner. Since E is usually given in pounds per square inch, it is advisable for the beginner to reduce all dimensions to pounds and inches.

Example I.—A soft steel wire, 150 yd. long, is subjected to a pull of 500 lb. The diameter of the wire is $d = 0.20$ in. Taking E as 30,000,000 lb./sq. in., find the total elongation of the wire.

The unit stress is

$$S = \frac{P}{A} = \frac{500}{\frac{\pi(0.20)^2}{4}} = 15{,}900 \text{ lb./sq. in.}$$

Hence the elastic limit (about 35,000 lb./sq. in. for soft steel) is not exceeded.

Therefore, by Eq. (1),

$$\lambda = \frac{PL}{AE} = \frac{SL}{E} = \frac{15{,}900 \times 150 \times 3 \times 12}{30{,}000{,}000}$$
$$= 0.286 \text{ in.}$$

Example II.—Solve Example I using the foot and ton as units. E must be expressed in tons per square foot. As given in Example I, $E = \dfrac{\text{pounds}}{\text{square inches}}$. Converting to tons per square foot,

$$E = \frac{\frac{\text{pounds}}{2000}}{\frac{\text{square inches}}{144}} = \frac{\text{pounds}}{\text{square inches}} \times \frac{144}{2000}$$
$$= \frac{30{,}000{,}000 \times 144}{2000} = 2{,}160{,}000 \text{ tons/sq. ft.}$$

Then

$$\lambda = \frac{PL}{AE} = \frac{0.25 \times 450}{\frac{\pi}{4}\left(\frac{0.20}{12}\right)^2 \times 2{,}160{,}000} = 0.0239 \text{ ft.} = 0.286 \text{ in.}$$

Example III.—A 10-in. length of steel measuring tape had a cross-section of 0.128 by 0.0127 in. (or 0.00162 sq. in.). The elongation under an applied load of 180 lb. was 0.0409 in. This was within the elastic limit of the piece. Find E.

$$E = \frac{PL}{A\lambda} = \frac{180 \times 10}{0.00162 \times (.0409)} = 27{,}150{,}000 \text{ lb./sq. in.}$$

54. Elongation of a Prism Due to Its Own Weight.—If a very long prism (such as a mine pump rod) is suspended from one end, the elongation of the prism due to its own weight must be considered. In Fig. 59*a*, the prism is shown in its unstrained condition. At a distance x from its lower end, take a typical portion or element ED whose length in the unstrained condition is dx. Owing to the weight of DO, the part of the prism below the section CD, the element ED is elongated by an amount $d\lambda$. If A is the sectional area and w is the heaviness (weight per cubic inch) of this prism, the weight of part DO pulling down on the lower end of the element ED is $P = wAx$. In Fig. 59*b* the prism is shown in its strained condition and the length of the element ED after strain is shown as $dx + d\lambda$.

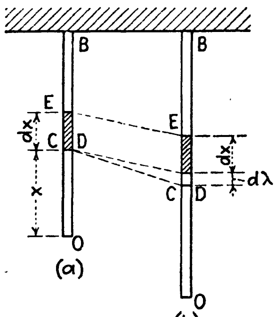


FIG. 59.

Since the elongation of a prism of length L and sectional area A resulting from an end load P is $\lambda = PL/AE$ (within the elastic limit), we may write $d\lambda$ for λ, dx for L, and wAx for P.

Therefore

$$d\lambda = \frac{wAxdx}{AE}.$$

This is the elongation of any typical element of the prism due to the weight of the part below that element.

The total elongation of the entire prism of length L will be the sum of the elongations of its elemental portions.

Therefore

$$\lambda = \frac{\int_0^L wAxdx}{AE} = \frac{wAL^2}{2AE} = \frac{GL}{2AE}, \tag{3}$$

where $G = wA\,L$ = weight of prism. By comparing the equations $\lambda = PL/AE$ and $\lambda = GL/2AE$, it is clear that *the elongation of a prism due to its own weight is equivalent to the elongation of the same prism due to an end load of one-half the weight of the prism; or it is equivalent to the whole weight applied to the half length.*

The maximum intensity of the tensile stress in the prism due to its own weight occurs at the upper end B and is

$$S_{\max.} = \frac{G}{A}.$$

Example.—A steel wire, 2880 ft. long, is suspended in the vertical shaft of a mine, and a weight of $P = 500$ lb. is hung from its lower end. The sectional area of the wire is

$$A = 0.10 \text{ sq. in.}$$

Taking $w = 490$ lb./cu. ft. and $E = 30{,}000{,}000$ lb./sq. in., find the total elongation and the maximum intensity of stress in the wire.

The elongation due solely to the end load P is

$$\lambda_1 = \frac{500 \times 2880 \times 12}{0.10 \times 30{,}000{,}000} = 5.76 \text{ in.}$$

The weight of the wire is

$$G = wAL = 490 \times \left(\frac{0.10}{144}\right) \times 2880 = 980 \text{ lb.}$$

The elongation of the wire due to its own weight is

$$\lambda_2 = \frac{980 \times 2880 \times 12}{2 \times (0.10) \times 30{,}000{,}000} = 5.65 \text{ in.}$$

The total elongation is, therefore,

$$\lambda = \lambda_1 + \lambda_2 = 11.41 \text{ in.}$$

In this case, the elongation due to the wire's own weight is very nearly as much as that due to the end load P.

To find out whether or not the elastic limit has been exceeded, compute the maximum intensity of stress in the wire.

$$S_{\max.} = \frac{P + G}{A} = \frac{500 + 980}{0.10} = 14{,}800 \text{ lb./sq. in.}$$

which is well within the elastic limit for steel.

55. Temperature Stresses.—With few exceptions, the linear dimensions of a piece of material increase as the temperature rises and decrease as the temperature falls. If, then, a piece of material is prevented from expanding or contracting as the temperature rises or falls, stresses called *temperature stresses* will be induced in the material. To illustrate, Fig. 60 represents the steel arch of a bridge. This arch is firmly imbedded in concrete at O and B and these supports are assumed to be immovable. Suppose that the arch was put in place when the temperature of

the air was 50°F. As the temperature of the air rises (say to 100° in the summer), the arch tends to expand but is prevented from doing so by the immovable supports O and B. Hence temperature stresses are induced in the arch. As the temperature falls below 50° (let us say to $-20°$ in the winter), the arch tends to contract. The temperature stresses thus set up in the arch as the temperature either rises or falls may be appreciable and are generally considered in designing an arch. They are computed separately and are then combined algebraically with the stresses computed for the loading.

It should be noted that temperature stresses are induced only when the material is prevented from expanding or con-

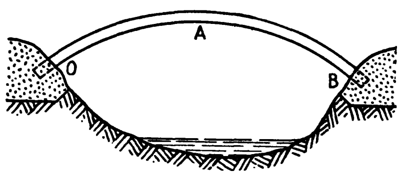


FIG. 60.

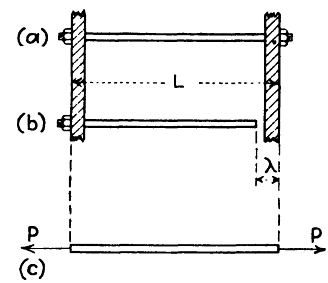


FIG. 61.

tracting as the temperature changes. In an ordinary railway bridge one end of the bridge rests on rollers or slides upon a plate. Such devices permit the bridge to expand or contract. This minimizes the temperature stresses.

Within ordinary limits of temperature change, the elongation or contraction of a material per unit length for 1° change of temperature is called the *coefficient of expansion* (or contraction) of the material and will be designated by η (Greek eta). If a steel rod of length L elongates (or contracts) by an amount λ due to a change of temperature $\Delta t° = t'' - t'$, then

$$\eta = \frac{\text{elongation per unit length}}{\text{change of temperature}} = \frac{\lambda}{L} \div \Delta t = \frac{\lambda}{L\Delta t}.$$

Or,

$$\lambda = \eta L \Delta t = \eta L(t'' - t') \tag{a}$$

in which t'' = final temperature.

t' = initial temperature.

Consider the rod shown in Fig. 61a of length L and sectional area A, whose ends are held fixed. This state may be brought

about by bolting the ends of the rod to immovable supports. It is required to find the tension in the rod due to a fall of temperature $\Delta t°$.

If the ends were free to move (not held fixed, Fig. 61*b*) the rod would contract by an amount $\lambda = \eta L \Delta t$. The tension in the rod with ends held fixed must be equal to the pull required to elongate the rod by the same amount (Fig. 61*c*). If P is the pull or tension in the rod (Art. 53),

$$\lambda = \frac{PL}{AE}. \tag{b}$$

Equating the two expressions for λ [Eqs. (*a*) and (*b*)],

$$\eta L \Delta t = \frac{PL}{AE}.$$

Therefore

$$P = \eta AE \Delta t. \tag{4}$$

This is the value for the tension in the rod if the ends are fixed and the temperature falls, provided the elastic limit is not exceeded. This equation for P does not contain the length L. Hence the temperature stress in the rod is independent of the length but is not independent of its sectional area A.

Note.—For values of η for some of the structural materials commonly used, see Appendix.

Example.—A steel rod of length $L = 30$ ft. and of sectional area $A = 0.50$ sq. in. is to be used to pull toward each other the two walls of a building (Fig. 61*a*). The temperature of the air (and rod) is 20°C. By means of torches the temperature of the rod is raised to 80°C., and the nuts on the ends of the rod are screwed up to a bearing. The rod is then allowed to cool to 20°C. Taking $\eta = 0.0000117$ and $E = 30{,}000{,}000$ lb./sq. in., what is the maximum pull which the rod can exert on the walls? $P = \eta AE\Delta t = 0.0000117 \times 0.50 \times 30{,}000{,}000 \times 60 = 10{,}530$ lb.

The unit stress in the rod is

$$S = \frac{P}{A} = \frac{10{,}530}{0.50} = 21{,}060 \text{ lb./sq. in.}$$

which is within the elastic limit. Should the walls yield before the temperature of the rod becomes 20°C., the tension in the rod will not reach this amount.

56. Statically Determinate and Statically Indeterminate Structures.—When all the unknown forces acting on a structure or

when all the stresses in all of its parts or members can be found from the equations of equilibrium alone, the structure is said to be *statically determinate.* The trusses considered in Elementary Statics were statically determinate since all the reactions acting on a truss and all the tensions and compressions in the members could be found from the equations of equilibrium, $\Sigma F_x = 0$; $\Sigma F_y = 0$; and Σmoms. $= 0$.

Whenever the equations of equilibrium will not suffice to determine all of the forces acting on a structure or all of the stresses in its parts or members, the structure is said to be *statically indeterminate.* To find those stresses or forces which are statically indeterminate, one or more equations in addition to the equations of equilibrium will be required. In a great many cases, such additional equations may be obtained in a manner similar to that used in the two articles which follow.

57. Compound Prism.—If two or more prisms, each of length L, are placed side by side and fastened together, the resulting

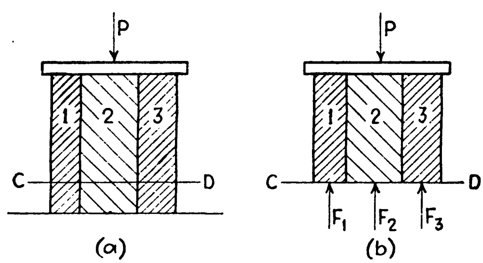


FIG. 62.

prism is called a *compound prism.* To investigate a compound prism consisting of three prisms (Fig. 62), proceed as follows: Assume that a longitudinal force P is applied in such a way that the three prisms must contract equally. If the three prisms are all made of the same material, the compound prism may be treated as a solid under simple compression. If, however, the three prisms are made of different materials, they will have, as a rule, different moduli of elasticity. Hence the unit stresses in the three prisms need not be the same even though they all contract equally.

Let A_1 be the sectional area of prism 1 and let E_1 be its modulus of elasticity. Similarly for prisms 2 and 3. Let CD be a right section of the compound prism and take "free" the part above CD (Fig. 62*b*). If F_1, F_2, and F_3 are the total stresses or forces

acting in the three prisms, respectively, it follows from statics that

$$F_1 + F_2 + F_3 = P. \tag{a}$$

In this case, statics furnishes only one equation. Since there are three unknowns, two more equations are required.*

By assumption, the three prisms contract equally. That is, $\lambda_1 = \lambda_2 = \lambda_3$.

Hence

$$\frac{F_1L}{A_1E_1} = \frac{F_2L}{A_2E_2} = \frac{F_3L}{A_3E_3}. \tag{b}$$

Equations (*a*) and (*b*) suffice to determine F_1, F_2, and F_3.

The unit stresses in the three prisms are, respectively,

$$S_1 = \frac{F_1}{A_1}, \qquad S_2 = \frac{F_2}{A_2} \qquad \text{and} \qquad S_3 = \frac{F_3}{A_3}.$$

Example.—Two steel plates each 1 by 4 in. are fastened to two opposite sides of a wooden prism 4 by 8 in. in size (Fig. 63). A longitudinal force of $P = 96{,}000$ lb. is applied to the compound prism. Assuming that the steel plates and the wood elongate equally, find the unit stress induced in the steel and also in the wood.

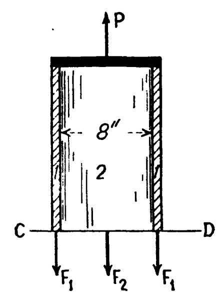


FIG. 63.

Let A_1, E_1, F_1, and S_1 refer to a steel plate (the two plates being alike), and A_2, E_2, F_2, and S_2 to the wooden prism. Take $E_1 = 30{,}000{,}000$ lb./sq. in. and $E_2 = 1{,}500{,}000$ lb./sq. in. The equations of statics determine one relation. That is, $\Sigma Y = 0$ gives

$$2F_1 + F_2 = 96{,}000. \tag{c}$$

Since the elongations of the three parts are equal,

$$\frac{F_1L}{A_1E_1} = \frac{F_2L}{A_2E_2} \qquad \text{or} \qquad \frac{F_1}{4 \times 30{,}000{,}000} = \frac{F_2}{32 \times 1{,}500{,}000}.$$

Therefore

$$F_1 = 2.5F_2. \tag{d}$$

Solving Eqs. (*c*) and (*d*),

$$F_1 = 40{,}000 \text{ lb.} \qquad \text{and} \qquad F_2 = 16{,}000 \text{ lb.}$$

* The use of $\Sigma M = 0$ will not help to find F_1, F_2, or F_3 since it will introduce another unknown, *viz.*, the distance to the point of application of P.

Finally,

$$S_1 = \frac{F_1}{A_1} = 10{,}000 \text{ lb./sq. in.} \qquad \text{and} \qquad S_2 = \frac{F_2}{A_2} = 500 \text{ lb./sq. in.}$$

These values of S are within the elastic limits.

58. Redundant Member.—Let ON (Fig. 64), be a rigid bar, B_1 a steel rod, and B_2 a wrought-iron rod. Neglect the weights of the bar and the rods. Assume B_1 and B_2 to have originally a close fit. That is, before the load Q is applied, B_1 and B_2 are just taut but not stretched. As soon as the load Q is applied, B_1 and B_2 are stressed and therefore strained (elongated).

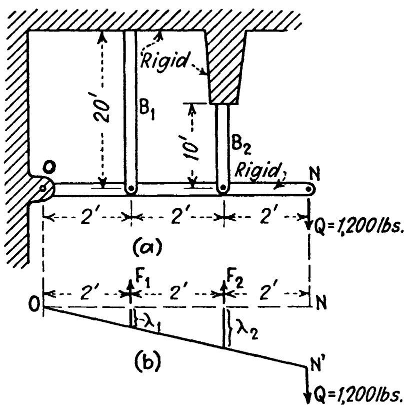


FIG. 64.

It is evident that, as a matter of stability, only one of the rods is necessary. If B_1 is removed, B_2 (if strong enough) would sustain the loaded bar ON. So one of the rods is *redundant*.

With both rods in place and the load Q applied, the total stress in each rod will depend upon the relative ease with which these rods can be stretched. If B_1 is a very slender rod compared with B_2, the total stress in B_1 might be negligible with respect to the total stress in B_2. Again, given a steel rod and a wrought-iron rod of the same size and length, the wrought-iron rod will be stretched more easily than the steel rod. Consequently, the total stresses in the rods B_1 and B_2 depend not only upon the laws of statics but also upon the elastic properties of the rods.

Example. Fig. 64.—Given, for the steel rod B_1, $L_1 = 20$ ft., $A_1 = 0.40$ sq. in., and $E_1 = 30{,}000{,}000$ lb./sq. in.; and, for the wrought-iron rod B_2, $L_2 = 10$ ft., $A_2 = 0.10$ sq. in., and $E_2 = 25{,}000{,}000$ lb./sq. in. It is required to find the total

stresses F_1 and F_2 in these rods if $Q = 1200$ lb. Other dimensions are shown in the figure.

Consider the bar free (Fig. 64b). Let ON be the original position and ON' the final position of the bar. The bar itself is assumed not to bend; *i.e.*, it is so stiff that its bending may be assumed to be negligible when compared with the elongations of the rods. If λ_1 is the elongation of B_1 and λ_2 is the elongation of B_2, it is evident from the geometry of the figure that $\lambda_2 = 2\lambda_1$. Hence

$$\frac{F_2L_2}{A_2E_2} = 2\frac{F_1L_1}{A_1E_1}.$$

Or, substituting values for A_1, A_2, L_1, L_2, E_1 and E_2,

$$\frac{F_2 \times 10 \times 12}{0.10 \times 25{,}000{,}000} = \frac{2F_1 \times 20 \times 12}{0.40 \times 30{,}000{,}000}.$$

Simplifying,

$$F_2 = \frac{5F_1}{6}. \tag{a}$$

This equation contains two unknowns, F_1 and F_2. Another equation is necessary. Statics furnishes this additional equation. $\Sigma M_O = 0$ gives

$$1200 \times 6 - 4F_2 - 2F_1 = 0. \tag{b}$$

Solving Eqs. (a) and (b),

$$F_1 = 1350 \text{ lb.} \qquad \text{and} \qquad F_2 = 1125 \text{ lb.}$$

The unit stresses are

$$S_1 = \frac{F_1}{A_1} = \frac{1350}{0.40} = 3875 \text{ lb./sq. in.}$$

and

$$S_2 = \frac{F_2}{A_2} = \frac{1125}{0.10} = 11{,}250 \text{ lb./sq. in.}$$

Evidently, the elastic limits are not exceeded. It may be added that the other two equations of equilibrium, $\Sigma X = 0$ and $\Sigma Y = 0$, introduce the reactions at O and that, therefore, after F_1 and F_2 have been found the two unknown components at O may be computed.

23. Fig. 65.—A rigid bar is supported by three rods each one in a vertical position as shown in the figure. The two rods B_1 and B_1 are steel rods for which $L_1 = 100$ ft., $A_1 = 0.50$ sq. in., and $E_1 = 29{,}000{,}000$ lb./sq. in.

B_2 is a copper rod with $L_2 = 100$ ft., $A_2 = 2$ sq. in., and $E_2 = 8{,}000{,}000$ lb./sq. in. Find the total stress in each of the rods.

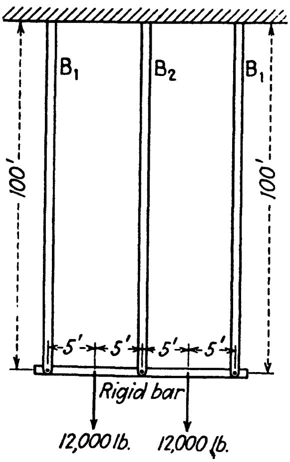


FIG. 65.

Owing to the symmetrical arrangement of rods and loads, the three rods will elongate equally.

Ans. $F_1 = 7{,}725$ lb.; $F_2 = 8{,}550$ lb.

WORK AND ENERGY. RESILIENCE. IMPACT

59. External and Internal Work.—When a body is deformed (stretched, bent, twisted, etc.), the points of application of the external forces producing the deformation move. Hence work is done by the external forces. The work done by the external forces acting on a body is conveniently called the *external work.*

When a body is deformed, work must be done to deform the elements or parts of that body. This work is conveniently called the *internal work.*

Now work is the measure of energy expended. Hence, if all the energy expended by the external forces is used to deform the body, it follows from the law of the conservation of energy that

$$\text{External work} = \text{internal work.}$$

60. Force or Load Gradually Applied.—According to Hooke's law, if a prism is acted upon by an axial force the elongation of the prism is proportional to the force acting, provided the elastic limit is not exceeded. Obviously, the law fails if the force is so suddenly applied that part of the force must be used to overcome the inertia of the particles of the prism or if an appreciable part of the energy expended by the external force (or forces) is transformed into kinetic energy of the particles.

When tensile tests are made, the force, as a rule, is made to increase gradually from zero to its final value and is applied so slowly that the inertia of the particles may be neglected at all stages in the elongation. A load thus applied is called a *force or load gradually applied.* Hence Hooke's law holds if the force is gradually applied.

In many cases, the force, although not gradually applied, is applied in such a way that the final velocities of the particles are zero (momentarily). In such cases, in considering the whole range of motion, the total change in kinetic energy is zero (initial velocities of particles assumed to be zero) and the assumption may be made that the total energy expended by the external force (or forces) is used to deform the prism.

In the problems to be considered, it will be assumed that the inertia of the particles of the prism may be neglected and that therefore the force may be treated as if gradually applied.*

61. Work Done in Stretching a Prism under Simple Tension.—Let the prism be acted upon by an axial force that gradually

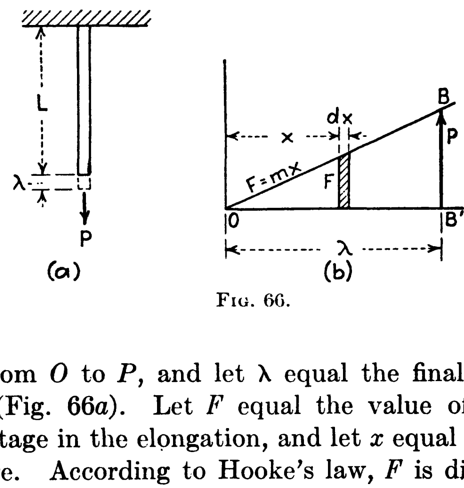


FIG. 66.

increases from O to P, and let λ equal the final elongation of the prism (Fig. 66*a*). Let F equal the value of this force at any given stage in the elongation, and let x equal the elongation at this stage. According to Hooke's law, F is directly proportional to x, provided the elastic limit is not exceeded. Under these circumstances,

$$F = mx$$

where m is a constant. Hence, within the elastic limit, the relation between F and x as F increases gradually from zero value to

* If a sledge hammer is used to strike (say) a beam resting on end supports, the hammer may rebound before the beam as a whole is appreciably deformed. Moreover, in the neighborhood of the point where the hammer hits there may be a local deformation of such magnitude as to injure the beam near that point. That is, the maximum stress in the beam in the neighborhood of the point where the hammer hits may be larger than that resulting from the deformation of the beam as a whole. Such localized stresses require special consideration.

the value P may be represented by the straight line OB (Fig. 66*b*).

The area of $\triangle OBB' = P\lambda/2$. By calculus, the area of $\triangle OBB'$ is $\int_0^\lambda F dx$. Now Fdx is the work done by F as the prism is elongated an infinitesimal amount dx. The total work done by F as F increases gradually from O to P is therefore,

$$U = \int_0^\lambda F dx = \text{area of } \triangle OBB' = \frac{P\lambda}{2}.$$

Hence, if P is a load gradually applied, and λ equals total elongation (or contraction) produced, then, within the elastic limit, the total work done is

$$U = \frac{P\lambda}{2}. \tag{5}$$

Equation (5) may be put in another form. If A is the sectional area of the prism and S is the final unit stress induced in the prism by the force P, then $P = AS$. Also, within the elastic limit,

$$\lambda = \frac{PL}{AE} = \frac{SL}{E}.$$

Hence

$$\frac{P\lambda}{2} = \frac{AS}{2} \times \frac{SL}{E} = \frac{S^2AL}{2E}.$$

Within the elastic limit, the total work done in stretching the prism is, therefore,

$$U = \frac{S^2}{2E} \times AL = \frac{1}{2}\frac{S^2}{E} \times \text{volume of prism.} \tag{6}$$

Example.—A steel rod of length $L = 5$ ft. and of sectional area $A = 0.50$ sq. in. is stretched so that the unit stress is $S = 18{,}000$ lb./sq. in. The modulus of elasticity is

$$E = 30{,}000{,}000 \text{ lb./sq. in.}$$

How much work is done in stretching the prism?

$U = \frac{1}{2}\frac{18{,}000^2}{30{,}000{,}000} \times (0.5 \times 5 \times 12) = 162$ in.-lb. of work $=$ 13.5 ft.-lb.

62. Note.—When a prism is stretched beyond its elastic limit, the law of proportionality no longer holds. That is, the relation between F, the force, and x, the elongation produced, is no

longer linear. In Fig. 67, let OA represent the relation between F and x up to the elastic limit, and AB that beyond the elastic limit. It is still true that the total work done in stretching the prism is

$$U = \int_0^\lambda F dx$$

and that this work is represented by the total area under the curve OAB. Note, however, that the area under the curve OAB does *not* equal $P\lambda/2$. Hence Eq. (5) or Eq. (6) should not be used when the elastic limit is exceeded.

63. Weight Suddenly Applied to a Prism.—Given a prism that is supplied with a flange at its lower end (Fig. 68). Let L equal length of prism in its unstretched state. If a body whose weight is W (body not shown in figure) is held (say, by the hands)

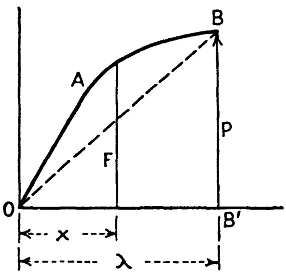


FIG. 67.

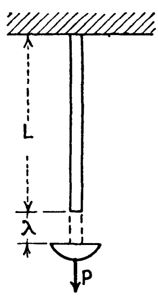


FIG. 68.

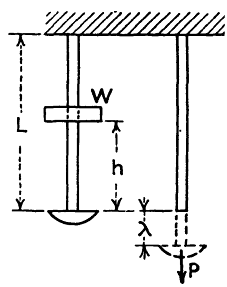


FIG. 69.

so that it just touches the flange of the unstretched prism and is then suddenly released (hands suddenly removed), the body begins to sink and the downward pressure exerted by the body on the flange increases from zero to some value P.

If λ equals the elongation of the prism when the weight comes to rest (momentarily) and is the maximum elongation of the prism, then, within the elastic limit, the work done by P in stretching the prism is (Art. 61)

$$U = \frac{P\lambda}{2}.$$

On the other hand, the body whose weight is W sinks through a distance λ. Hence the work done by W (*i.e.*, by gravity) is

$$U = W\lambda.$$

If it is assumed now that all the energy expended by W is used to stretch the prism (no change in kinetic energy), it follows from

the law of the conservation of energy that, within the elastic limit,

$$W\lambda = \frac{P\lambda}{2}.$$

Or $P = 2W$ is the final force exerted on the rod. Hence, *within the elastic limit, a weight suddenly applied to the prism* produces double the tension and therefore double the stress it would produce if gradually applied.*

If the body drops through a distance h before it strikes the flange of the prism (Fig. 69), the work done by W (gravity) is

$$U = W(h + \lambda).$$

On the other hand, within the elastic limit, the work done by P in stretching the prism is

$$U = \frac{P\lambda}{2}.$$

Hence, within the elastic limit,

$$\frac{P\lambda}{2} = W(h + \lambda)$$

or

$$P = \frac{2W(h + \lambda)}{\lambda}.$$

If λ is small when compared with h, the last equation may be written

$$P = \frac{2Wh}{\lambda}.$$

Note.—It is assumed in the preceding argument that the falling body or the flange does not deform appreciably; and that the support does not move. If, for instance, the flange should bend, part of the energy expended by the falling body would be used to bend the flange.

* If the load W is to be gradually applied, the body must be held (say by the hands) and then gradually released. As the body is being released, it slowly sinks (velocity negligible). The hands support less and less of the weight and the flange more and more until finally the flange supports the whole weight. That is, finally, $P = W$. In the gradual sinking of the weight, one-half of the work done by W is done to offset the gradually decreasing upward pressure of the hands and the other half is done to overcome the gradually increasing upward pressure of the flange. That is, only half of the energy expended by W is used to stretch the prism.

64. Force Suddenly Reversed.—A force Q is gradually applied to the end of a rod, producing an elongation λ (Fig. 70). The force Q is then suddenly reversed. This is an ideal case but one suggestive of the magnified effect that may be caused by vibrations, especially if reversals of stress are produced in a member.

The energy stored up in the rod by the gradual application of the force Q is $Q\lambda/2$. When the load is suddenly reversed, this energy, together with the work done by the reversed force Q through the distance $\lambda + \lambda'$, will be expended in compressing the rod a distance λ', and the total energy stored up in the rod will be $P\lambda'/2$. That is (initial and final velocities being zero),

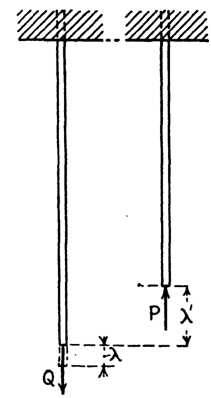


FIG. 70.

$$\frac{Q\lambda}{2} + Q(\lambda + \lambda') = \frac{P\lambda'}{2}.$$

Or

$$\frac{3Q\lambda}{2} + Q\lambda' = \frac{P\lambda'}{2}.$$

Now $\lambda'/\lambda = P/Q$ within the elastic limit. So

$$\frac{3Q\lambda}{2} + P\lambda = \frac{P^2\lambda}{2Q},$$

from which λ cancels. Then $3Q^2 + 2PQ = P^2$. Completing the square, taking P as the unknown,

$$P^2 - 2PQ + Q^2 = 3Q^2 + Q^2 = 4Q^2,$$
$$P - Q = 2Q.$$

Therefore

$$P = 3Q.$$

So the effect of a force suddenly reversed is three times the effect of the same force gradually applied.

65. Resilience.—So long as the stresses produced in a body are within the limits of elasticity, the body will resume its original size and shape when the external forces producing the strain are gradually removed. Hence the energy expended upon a body in producing an *elastic strain* is stored up in that body as *potential energy* and this energy is returned when the body recovers. For instance, within the elastic limit, the energy expended in winding a watch spring is stored up in that spring in the form of potential energy and this energy is returned by the spring as it runs the watch.

When the elastic limit is exceeded and the material is overstrained, part of the energy expended in producing the strain is used in producing a permanent set and the rest is stored up in

that body as potential energy. That is, an overstrained body is not able to return all the energy expended upon it in producing the strain.

To illustrate, let OAH (Fig. 71) be the stress-strain diagram for a steel rod stretched beyond its elastic limit. At the stage H let the stress be gradually removed. The stress-strain diagram for the return motion is given by HJ. That is, the rod will not resume its original size when the stress is removed. Now the work done (per unit volume of the rod) in stretching

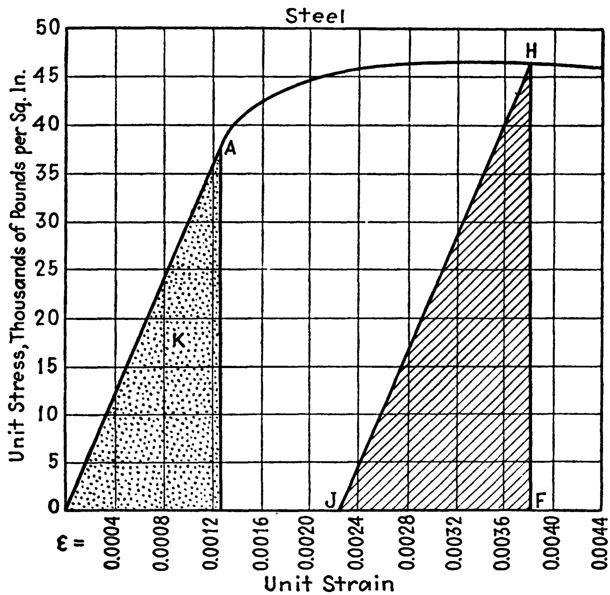


FIG. 71.

the rod up to H is represented by the area $OAHF$ (area under the curve OAH, see Art. 62), and that done by the rod in its recovery is represented by the area JHF. Hence the potential energy (per unit volume) stored up in the rod as the rod is stretched up to H is represented by the area JHF and the energy used to deform permanently the rod is represented by the area $OAHJ$.

The potential energy stored up in a body when under strain is called *resilience*. The resilience of a strained body, then, is the work this body can do by virtue of its elasticity.

The resilience of a body (potential energy stored up in the body) when stressed up to its elastic limit is called *elastic resilience*

and is numerically equal to the total work done by the external force or forces producing the strain.

The elastic resilience of a member is a matter of practical importance. For a prism under simple tension, the elastic resilience is [Art. 61, Eq. (6)]

$$U = \frac{1}{2}\frac{S_e^2}{E} \times \text{volume of prism} \tag{7}$$

where S_e = elastic limit.

Example.—Compare the elastic resilience of two soft-steel rods. One is 1 sq. in. in area and 3 in. long, and the other is 0.50 sq. in. in area and 12 in. long. Take the elastic limit for soft steel as $S_e = 30{,}000$ lb./sq. in.

For the shorter rod,

$$U_1 = \frac{S_e^2 AL}{2E} = \frac{\overline{30{,}000}^2 \times 1 \times 3}{2 \times 30{,}000{,}000} = 45 \text{ in.-lb.}$$

For the longer rod,

$$U_2 = \frac{\overline{30{,}000}^2 \times \frac{1}{2} \times 12}{2 \times 30{,}000{,}000} = 90 \text{ in.-lb.}$$

The long rod can absorb more energy of shock than the short rod. Conversely, if a given amount of energy must be absorbed, the unit stress induced in the long rod will be less than that in the short rod.*

66. Modulus of Resilience.—Parts of machines and structures often are subject to shocks or impact. It may be necessary, therefore, to design a member on the basis of its ability to absorb energy, *i.e.*, on the basis of resilience. When that is necessary, it is convenient to know what the elastic resilience of unit volume of a material is. The elastic resilience of a material per unit volume is called *modulus of resilience* and will be designated by K. From Eq. 7 (Art. 65), it is seen that for a prism in tension the modulus of resilience is

$$K = \frac{1}{2}\frac{S_e^2}{E}$$

where S_e = elastic limit.

* See also p. 271 of Prof. F. B. Seely's "Resistance of Materials" for interesting examples of constructions for increasing resistance to shock, or for absorbing more energy without exceeding a specified stress.

Note that in Fig. 71 the dotted area K represents the modulus of resilience of the rod.

Illustration.—If for structural steel $S_e = 35{,}000$ lb./sq. in. and $E = 30{,}000{,}000$ lb./sq. in.

$$K = \frac{\overline{35{,}000}^2}{2 \times 30{,}000{,}000} = 20.4 \text{ lb./sq. in.}$$

That is, a cubic inch of structural steel can absorb 20.4 in.-lb. of energy without exceeding its elastic limit.

67. Energy Load.—If a member is subjected to an impact load, the maximum force exerted on the member may be taken as an equivalent static load and the stress in the member or the deflection of that member may be calculated in the usual way. For instance, if a load W is suddenly applied to the flange of a rod (Art. 63), the stress or the elongation may be found by considering the rod as subjected to an equivalent static load, $P = 2W$. This assumes that the elastic limit is not exceeded.

Frequently, it is advantageous to attack a problem from the standpoint of the energy input or *energy load* which a structure or a member must absorb. To illustrate, if a weight W is allowed to fall freely through a distance h before it strikes the flange of the rod (Fig. 69), the energy expended by W (gravity) is Wh. That is, Wh equals energy load.* If all this energy is used to stretch the prism, we may put

$$Wh = \frac{1}{2}\frac{S^2AL}{E}.$$

Frequently, the impact is not due to a falling body. For instance, the prism may be horizontal and the body of weight W may be thrown against the flange. If V is the velocity of the body just before it strikes the flange, the kinetic energy of the body is $\frac{1}{2}MV^2$ where $M = W/g$. That is, the energy load is $\frac{1}{2}MV^2$. Hence

$$\frac{1}{2}MV^2 = \frac{1}{2}\frac{S^2AL}{E}.$$

In general, the energy load equals the total amount of energy the prism must absorb.

Example I. Fig. 69.—A weight of 4 lb. is dropped through a vertical distance of 5 ft. before it strikes the flange of a rod that

* If λ is too large to be neglected, the energy load is $W(h + \lambda)$, or, since $\lambda = SL/E$, the energy load $= W(h + SL/E)$.

is 10 ft. long. Find the minimum sectional area of the rod if the unit stress is not to exceed 15,000 lb./sq. in.

$$E = 30{,}000{,}000 \text{ lb./sq. in.}$$

$$U = Wh = \frac{S^2AL}{2E}.$$

Or

$$4 \times (5 \times 12) = \frac{(15{,}000)^2A \times 120}{2 \times 30{,}000{,}000}.$$

Therefore

$$A = 0.533 \text{ sq. in.}$$

Example II.—The length of an unstretched steel rod is

$$L = 5 \text{ ft.},$$

and its sectional area is $A = 2$ sq. in. With what velocity V may a body weighing $W = 64.4$ lb. strike the flange if the elastic limit (35,000 lb./sq. in.) is not to be exceeded? Note that

$$g = 32.2 \text{ ft./sec.}^2 = 32.2 \times 12 \text{ in./sec.}^2$$

$$U = \frac{1}{2}MV^2 = \frac{S^2AL}{2E}.$$

Therefore

$$\frac{1}{2} \times \frac{64.4}{32.2 \times 12}V^2 = \frac{(35{,}000)^2 \times 2 \times 5 \times 12}{2 \times 30{,}000{,}000}.$$

Or

$$V = 171.5 \text{ in./sec.} = 14.3 \text{ ft./sec.}$$

68. Impact Stresses Experimentally Determined.—Impact stresses require special consideration. In certain simple cases, the stresses due to impact may be calculated. Frequently, however, it is not possible to do so for the reason that the values of the impact loads are not definitely known. In such cases, it may become necessary to determine the impact stresses experimentally. For instance, a train running over a bridge produces impact. To find the impact stress in a certain member of that bridge, attach to that member a self-registering extensometer. First, let the train run very slowly over the bridge (load gradually applied). The extensometer registers the maximum elongation λ for a certain length L of the member as the train runs slowly over the bridge. Knowing λ and L, the unit stress S (static) may be

found from the equation [Eq. (2), Art. 53]

$$\lambda = \frac{SL}{E}.$$

In like manner, S_1 (impact stress) may be found when the train runs over the bridge say at 60 m. p. h. Suppose now that $S_1/S = 1.75$. This means that the impact stress in that member for trains running at 60 m. p. h. is 75 per cent greater than the static stress. Hence, when a corresponding member of a similar bridge is designed, the static stress calculated for that member should be increased 75 per cent to take care of the impact.*

69. Toughness.—Consider now the work that must be done per unit volume of a prism so as to rupture it. This work is represented by the area *OBDG* under the stress-strain curve (Fig. 72). In the case of steel, although the ultimate stress for soft steel may be much lower than that for hard steel, yet the total area under the stress-strain curve for soft steel may be greater than that for hard steel. Hence it requires more work per unit volume to rupture a piece of soft steel. That is, soft steel, owing to its ductility, can absorb more energy per unit volume than hard steel. This is of considerable practical importance in the case of a structure (such as a bridge) that may be subjected to impact or shocks of such magnitude that the unit stress may run considerably beyond the elastic limit. For instance, in designing a bridge, the impact due to trains running over it is considered. Suppose, however, that a derailment of an excursion train should take place on a bridge. The impact now may be much greater than that considered when designing the bridge and it is important for the bridge to be able to absorb the shock even though the bridge or some of its members may be injured.

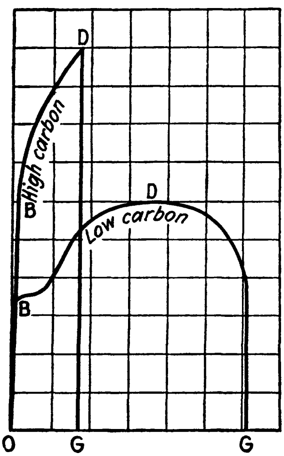


Fig. 72.

* Some authors apply the term "impact stress" to the stress in excess of the static stress. Accordingly, they would say that the impact stress is 75 per cent of the static stress and that the impact stress should be added to the static stress to obtain the stress to be used when designing that member.

The total amount of energy a material can absorb per unit volume before it is ruptured is a measure of the *toughness* of the material. Accordingly, soft steel is a tough metal. Likewise white oak and longleaf pine are tough woods. The *toughness of a material is represented by the total area under the stress-strain curve for that material.*

THIN-SHELLED CYLINDERS

70. Introduction.—When a hollow cylinder (a water pipe, for example) is subjected to an internal fluid pressure, stresses are developed in the *shell* or *wall* of the cylinder. Owing to the stresses thus developed, the cylinder tends to burst. To design a hollow cylinder, it becomes necessary, therefore, to know what stresses will be developed in its shell when the cylinder is subjected to a given internal fluid pressure.

If the thickness t of the shell or wall of a hollow cylinder is small when compared with the inner radius of the cylinder (Fig. 73), the cylinder is called a *thin-shelled* or a *thin-walled cylinder.* In the case of a thin-shelled cylinder, the stresses on a section of the shell may be assumed as uniformly distributed without introducing any appreciable error in the calculations. For instance, the stress on the transverse section of a pipe (shaded ring, Fig. 73) may be assumed as uniformly distributed provided t is small when compared with R. Only thin-shelled cylinders will be considered here. Water pipes, steam pipes, tanks, boilers, etc., as a rule, may be treated as thin-shelled cylinders.

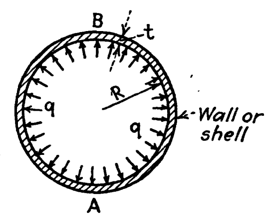


FIG. 73.

From the law of hydrostatics, it is known that the fluid pressure at any point on the inner surface of a hollow cylinder acts normally to the inner surface at that point. Note that in Fig. 73 the fluid pressure is represented as acting normally at all points on the inner surface and as having the same intensity q everywhere. If the weight of the fluid is considered, the fluid pressure at A will be slightly greater than at B. In practice q is taken as the average pressure, *i.e.*, as the intensity of the pressure at the center of the cylinder.

Note 1. *External Pressure.*—The stress induced in the shell depends, however, upon the difference between the inside and the

outside pressure. This difference is the "bursting pressure" and will be designated by q. That is, the cylinder will be represented as subjected to an internal pressure of q lb./sq. in.

Illustration.—If a boiler is subjected to an actual internal pressure of 164.7 lb./sq. in., and if the pressure of the outside air is 14.7 lb./sq. in., then $q = 164.7 - 14.7 = 150$ lb./sq. in. The boiler must be designed to withstand a net internal pressure of 150 lb./sq. in. If the internal (steam) pressure is measured by an ordinary gage, the "gage reading" gives the excess of the internal over the external pressure. It will be convenient at times to call q the *gage reading* or the *gage pressure.*

Note 2.—Sometimes a pipe is subjected to an external pressure that is greater than the internal pressure. This is the case, for

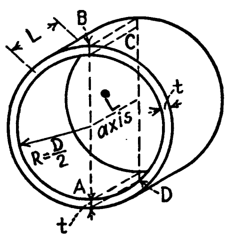


FIG. 74.

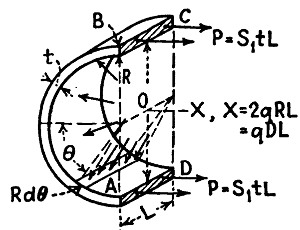


FIG. 75.

example, with cylindrical boiler flues. It may be the case also with a pipe under water. When the external pressure exceeds the internal pressure, a thin pipe (or any thin-walled hollow cylinder) tends to collapse; *i.e.*, it tends to bend or fold or cave in. Thin-shelled cylinders subjected to collapsing pressures require special consideration.*

71. Circumferential Stress.—Let L equal the length of a hollow cylinder between two transverse sections (Fig. 74). Imagine this cylinder divided into two halves (semicylinders) by means of the plane BD drawn vertically for the sake of convenience. Consider free the left half (Fig. 75). Note that the stress on a longitudinal section of this half shell, *i.e.*, on the narrow rectangle AD (or BC), acts circumferentially and therefore may be called the *circumferential stress* in the shell. Take the x-axis horizontal as shown. Now the forces that act on this half shell and that have components parallel to the x-axis are

1. The internal fluid pressure whose resultant will be designated by X. Evidently X acts parallel to the x-axis.

* See MAURER and WITHEY, "Strength of Materials," p. 80.

2. The tensile forces acting on the longitudinal sections AD and BC. If S_1 is the intensity of the stress on these sections (intensity of the circumferential stress), the total stress on each section is

$$P = S_1tL. \tag{a}$$

The half shell must be in equilibrium. Hence, putting $\Sigma F_x = 0$,

$$X = 2P.$$

That is [Eq. (*a*)],

$$X = 2S_1tL. \tag{b}$$

We must now determine X the resultant pressure on the inner surface of the half shell.

First Solution.—Consider an elemental longitudinal strip of length L and width $Rd\theta$ (Fig. 75). Its area is $dA = LRd\theta$. The total pressure on this strip is

$$dF = qdA = qLRd\theta,$$

and its horizontal component is

$$dF \cos \theta = qLR \cos \theta d\theta.$$

Now X equals the sum of the horizontal components of the pressure on the elemental strips beginning at A and ending at B. Therefore

$$X = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} qLR \cos \theta d\theta = qLR \sin \theta \Big]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} = 2qRL = qDL \tag{c}$$

where D = inner diameter = $2R$.

It is very important to note that the area of the diametral plane AC (Fig. 75) is $2RL$, or DL. Hence Eq. (*c*) becomes

$$X = q(2RL) = q(DL) = q \times (\text{area of diametral plane } AC).$$

That is, the *resultant pressure on the inner surface of a half shell equals the total pressure on a diametral plane of the hollow cylinder.*

To find S_1, the *intensity of the circumferential stress,* proceed as follows: Combining Eqs. (*b*) and (*c*), or directly from Fig. 75,

$$2S_1tL = 2qRL = qDL.$$

That is, the circumferential stress (stress on a longitudinal seam) is

$$S_1 = \frac{qR}{t} = \frac{qD}{2t}. \tag{8}$$

The intensity of the circumferential stress sometimes is called the *hoop tension.*

It is sometimes advantageous to find the total stress per unit length of seam. If T_1 equals the stress per unit length of longitudinal seam, $T_1 = S_1t$. Or [Eq. (8)]

$$T_1 = S_1t = qR = \frac{qD}{2}. \tag{9}$$

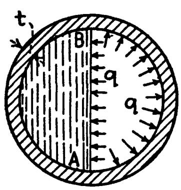


FIG. 76.

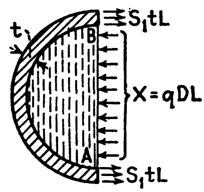


FIG. 77.

Second Solution.—Imagine the left half of the hollow cylinder solid (filled, say, with ice, Fig. 76), the intensity of the fluid pressure in the right half being q. Evidently this does not affect the stress in the shell at A or at B. Now take the left half free (Fig. 77).

From $\Sigma F_x = 0$,

$$2S_1tL = qLD \qquad \text{or} \qquad S_1 = \frac{qD}{2t} = \frac{qR}{t}.$$

72. Longitudinal Stress.—If the end of a hollow cylinder is closed and if this cylinder is subjected to an internal fluid pres-

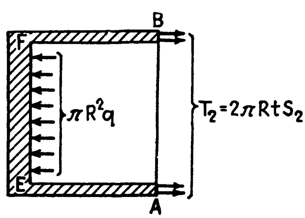


FIG. 78.

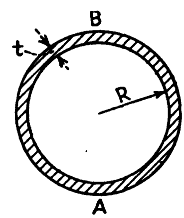


FIG. 79.

sure, stresses are developed in transverse sections of the cylinder giving rise to longitudinal stresses—longitudinal since such stresses are parallel to the axis of the cylinder.

Figure 78 shows the end of a closed thin-walled pipe or cylinder taken free (side view) and Fig. 79 shows the transverse section AB. The total pressure on the end EF of the cylinder equals

$\pi R^2 q$. If S_2 equals the intensity of the stress on the transverse section AB of the shell (intensity of the longitudinal stress), the total stress on $AB = 2\pi RtS_2$ (approximately).* Hence, for equilibrium to exist,

$$2\pi RtS_2 = \pi R^2 q.$$

That is, the intensity of the longitudinal stress (stress on a transverse or circumferential seam) is

$$S_2 = \frac{qR}{2t} = \frac{qD}{4t}. \tag{10}$$

This equation gives also the intensity of the stress on a central section of a thin-shelled *hollow sphere* under internal fluid pressure.

Note.—By comparing Eq. (10) with Eq. (8), it is seen that $S_1 = 2S_2$. That is, the intensity of the stress on a longitudinal section (circumferential stress) is twice that on a transverse section (longitudinal stress). Accordingly, if the water in a closed pipe freezes, the pipe will split along a seam that runs lengthwise.

Example I.—If a cast-iron water pipe 1 ft. in diameter ($D = 1$ ft.) is to be subjected to an internal pressure (gage pressure) of 200 lb./sq. in., what should be the minimum thickness of the shell if the unit tensile stress in the shell is not to exceed 4000 lb./sq. in.?

From Eq. (8),

$$t = \frac{qD}{2S_1} = \frac{200 \times 1 \times 12}{2 \times 4000} = 0.30 \text{ in.}$$

Example II.—A steam boiler is to be 4 ft. in diameter ($D = 4$ ft.). The boiler is to be subjected to a gage pressure of

$$q = 150 \text{ lb./sq. in.}$$

What will be the tension in the shell per linear inch of longitudinal seam? Per linear inch of circumferential seam?

Total tension per linear inch of longitudinal seam is [Eq. (9)]

$$T_1 = \frac{qD}{2} = \frac{150 \times 4 \times 12}{2} = 3600 \text{ lb.}$$

* The area of the transverse section of the shell (Fig. 79) is $2\pi\left(R + \frac{t}{2}\right)t$. In a thin-shelled cylinder, $t/2$ is small when compared with R and no appreciable error is made if the area of the transverse section is taken as $2\pi Rt$, *i.e.*, as equal to the inner circumference times the thickness of the shell.

and total tension per linear inch of circumferential seam is

$$T_2 = \frac{T_1}{2} = 1800 \text{ lb.}$$

PROBLEMS

24. A rod consists of two parts firmly connected at C (Fig. 80). The lower part is of brass (radius = 1.25 in., length = 80 ft., E = 14,000,000 lb./sq. in.). The upper part is of copper (diam. = 3 in., length = 250 ft., E = 15,000,000 lb./sq. in.). Compute the total elongation due to a weight of 42,000 lb. hung from the lower end. (Neglect the weight of the rod.)

Ans. λ_{cu} = 1.19 in.; λ_{br} = 0.59 in.; λ = 1.78 in.

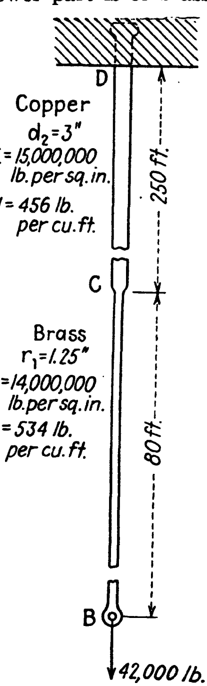


FIG. 80.

25. In Problem 24, assume that the end weight is removed. Compute the total elongation due to the weight of the rod. Brass weighs 534 lb./cu. ft. Copper weighs 456 lb./cu. ft.

Ans. W_{cu} = 5600 lb.; W_{br} = 1460 lb.; λ_{cu} = 0.12 in.; λ_{br} = 0.01 in.

26. In Problem 24, considering both the end load and the weight of the rod, find the maximum unit tensile stress induced in the rod. *Ans.* 6900 lb./sq. in. at D.

27. A steel wire has a sectional area of A = 0.15 sq. in. It is 450 ft. long when subjected to a pull of 1000 lb. What will be the length when it is subjected to a pull of 4000 lb.? *Ans.* 450 ft. 3.6 in.

28. A steel bar, $\frac{3}{4}$ in. square and 12 ft. long, is subjected to a pull of 9,000 lb. and the ends of the bar are then held fixed (fastened to immovable supports). How much must the temperature rise for the stress in the bar to become zero? Take the temperature coefficient as η = 0.000,0065. *Ans.* 82°F.

29. A steel rod, 16 ft. long, is rigidly held at its ends. If at 40°F. the tension in the rod is 10,000 lb./sq. in., what will be the stress at 90°F.? *Ans.* 250 lb./sq. in.

30. One end of a steel rod, area = $\frac{1}{2}$ sq. in. and 4 ft. long, is welded to one end of a copper rod, area = $\frac{3}{4}$ sq. in. and 6 ft. long. The other ends are fastened to immovable supports. At 100°F. the two rods are just taut but not stretched. What will be the total stress in each rod when the temperature is 20°F.? For copper, η = 0.000,0093 and E = 15,000,000 lb./sq. in.

Suggestion.—Assume the end B released. Find the total contraction of the rods due to temperature change. Then find the force P necessary to bring B back to its initial position. Find unit stresses.

Ans. P = 8180 lb.; S for steel, 16,360 lb./sq. in.; for copper, 10,900 lb./sq. in.

31. A timber, $5\frac{1}{2}$ by $5\frac{1}{2}$ in., has two $5\frac{1}{2}$ by $\frac{1}{4}$-in. steel plates bolted to opposite sides. The compound member is subjected to an axial force of 72,000 lb.

Find the unit stress in the steel and in the wood. For steel, $E = 30{,}000{,}000$ lb./sq. in.; for wood, $E = 1{,}500{,}000$ lb./sq. in.

Ans. 16,900 lb./sq. in.; 850 lb./sq. in.

32. A timber, 6 by 8 in., is reinforced on all four sides by plates, two being 6 by $\frac{1}{8}$ in. and the other two 8 by $\frac{1}{4}$ in. The allowable stress for steel is 20,000 lb./sq. in. and that for wood is 1200 lb./sq. in. Find the maximum axial load the compound member can safely carry. Let $n = E_s/E_w = 20$. Assume first that the steel governs; *i.e.*, assume that the stress in the steel is 20,000 lb./sq. in. and find P_1. Next assume that the wood governs and find P_2. The smaller value of the two is to be used. *Ans.* 158,000 lb.

33. Three steel rods are arranged in a vertical plane to support a 10-ton load, as shown in Fig. 81. Before the load was applied, the rods fitted closely but without initial stress. The central vertical rod (1) is 30 ft. long and has a sectional area of $A_1 = 0.80$ sq. in. Each of the oblique rods (2) has a sectional area of $A_2 = 0.25$ sq. in. Find the total stress in each rod.

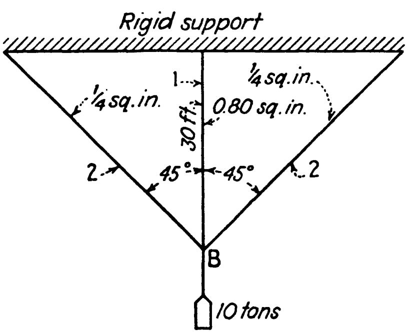


FIG. 81.

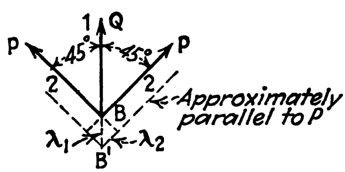


FIG. 82.

Suggestion.—The point B moves from B to B' (Fig. 82). λ_1 being small, we may assume the angles to remain 45°. From geometry, express λ_1 in terms of λ_2. *Ans.* $P = 2560$ lb.; $Q = 16{,}380$ lb.

34. Referring to Fig. 69, let the sectional area of the rod be $A = 0.60$ sq. in., $L = 14$ ft., and $W = 6$ lb. Through what distance may the weight be dropped if the unit stress in the rod is not to exceed 18,000 lb./sq. in.? (See Art. 67.) *Ans.* 7.56 ft.

35. A steel rod, 0.75 sq. in. in area and 10 ft. long, supports a total load of 4500 lb. How much energy will be stored up in the rod due to a gradual increase of the load from 4500 to 22,500 lb.? *Ans.* 1,296 in.-lb.

36. In Problem 35, what are the final total resilience and the unit resilience?

Ans. 1350 in.-lb.; 15 in.-lb./cu. in.

37. A water pipe is 25 ft. in diameter (about 79 ft. in circumference) and 500 ft. long. Its shell is 2.5 in. thick. Owing to the radial water pressure, the shell is under a tension of 18,000 lb./sq. in. ($S_1 = 18{,}000$ lb./sq. in., Fig. 75). Find the increase in the circumference and the decrease in the length of this pipe due to the radial water pressure, assuming the pipe free to contract longitudinally. Take $E = 30{,}000{,}000$ lb./sq. in. and m (Poisson's ratio, Art. 22) equal to 0.30. *Ans.* 0.570 in.; 1.08 in.

Note.—At the Boulder Dam, water pipes were used that are 30 ft. in diameter. The ends, however, are firmly anchored.

38. The pipe of Problem 37 is to be *firmly anchored at its ends* and is to be installed at such a temperature that at a normal temperature of 65°F. the longitudinal expansion of the pipe due to change of temperature will be equal to the longitudinal contraction due to the radial water pressure. In other words, under normal working conditions (pipe filled with water at 65°F.), the total longitudinal stress in the pipe is to be zero. At what temperature should the pipe be installed? Take the coefficient of temperature expansion as $\eta = 0.000,0065$. *Ans.* 37.7°F.

39. What will accordingly be the total tension and the unit tension in the pipe when empty and at normal temperature, the ends being firmly held?

Ans. 6414 tons, 5400 lb./sq. in.

40. A water pipe, 2.5 ft. in diameter, is subjected to an internal water pressure of $q = 300$ lb./sq. in. Find the tension per linear inch of longitudinal seam. If the seams are welded and as strong as the solid metal, find the thickness t of the shell if $S = 10,000$ lb./sq. in.

Ans. 4500 lb.; 0.45 in.

41. In the Hydraulic Laboratory at Cornell University there is a riveted steel standpipe 6 ft. in diameter and 60 ft. high. Assume that the vertical riveted seams have an efficiency of 62 per cent; *i.e.*, the strength of a seam is 0.62 times that of the solid plate. It is required to find the thickness of the lowest plates if $S = 16,000$ lb./sq. in. *Ans.* $t = 0.0945$ in.

Note.—Since the steel in a standpipe will rust, it is necessary to increase the thickness by $\frac{1}{8}$ to $\frac{1}{4}$ in. The lower plates of the actual pipe are $\frac{5}{16}$ or 0.3125 in. thick.

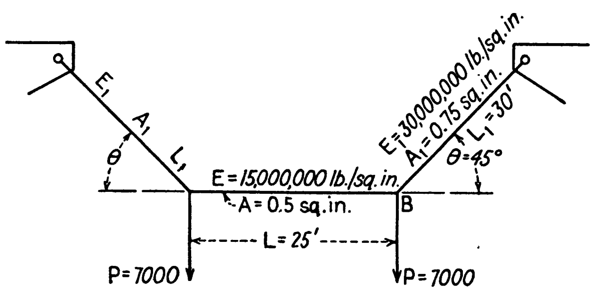


FIG. 83.

42. Three prismatic rods are fixed to unyielding supports and connected as shown in Fig. 83. The two inclined rods have equal values of angle with the horizontal, length, modulus of elasticity, and sectional area. These values are given in the figure. Neglect the weights of the rods. Find the elongation of each rod when each of the two loads (P, P) equals 7,000 lb.

Ans. $\lambda = 0.28$ in.; $\lambda_1 = 0.158$ in.

43. In Problem 42, find the vertical and the horizontal displacement of the point B, noting that the center of the horizontal rod moves vertically downward on a line of symmetry.

Ans. 0.31 in. down; 0.14 in. to the right.

CHAPTER IV

RIVETED JOINTS

73. Introduction.—The simplest riveted joint is made by lapping an edge or end of one plate over the end of another and fastening the two plates together by means of a single rivet (Fig. 84).

Rivets are driven while red hot. On cooling, a rivet contracts. As a result, the plates are pressed together sometimes with considerable force. Owing to the pressure between the plates there may be developed an appreciable resistance to sliding of plates.

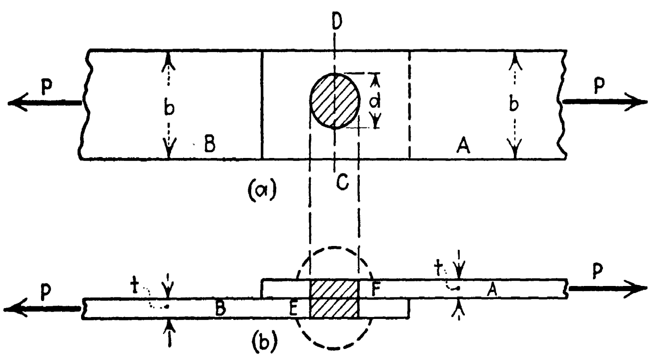


FIG. 84.

That is, the force may be transmitted from one plate to the other partly by friction. It has been found, however, that the plates may slip even when the joint is subjected to a normal load and that therefore friction between the plates cannot be depended upon as giving any material assistance in transmitting the force from one plate to the other. In practice, it has become customary *to neglect the frictional resistance between the plates and to assume that the entire force is transmitted from one plate to the other through the rivet.* This assumption will be made in the analyses of the joints considered in this chapter.

74. Stress in a Lap Joint with a Single Rivet. Fig. 84.—The total force (pull) acting on the joint is P.

Let b = width of plate.
t = thickness of plate.
d = diameter of rivet = diameter of rivet hole.

That is, it will be assumed that the rivet fills the hole.

a. Shear in Rivet.—Owing to the pull P, the rivet tends to shear along the section EF (Fig. 84b), *i.e.*, in the plane of contact of the two plates. Since shearing action takes place on only one cross-section, EF, the rivet is said to be in *single shear*. Figure 85 shows a sketch of a rivet that has partly failed in single shear.

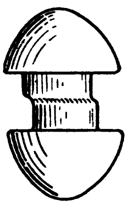


FIG. 85.

If A_s = sectional area of the rivet, and
S_s = safe unit shearing stress in the rivet,

then the maximum value of P consistent with safety against shearing of the rivet is

$$P_s = A_sS_s = \frac{\pi d^2}{4}S_s.$$

The value of P given by this equation is called the *safe* strength of the joint in shear and is designated by P_s.

If there are n rivets, each of diameter d, then

$$A_s = n\frac{\pi d^2}{4},$$

and

$$P_s = A_sS_s = n\frac{\pi d^2}{4}S_s.$$

b. Bearing between Rivet and Plate.—Consider the plate shown in Fig. 86. The pull in the plate is balanced by the pressure

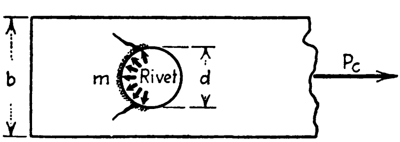


FIG. 86.

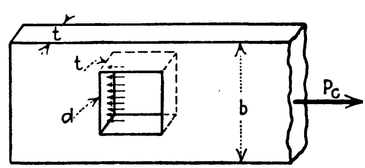


FIG. 87.

of the rivet against the side of the hole in the plate as the sketch shows. If the intensity of this pressure is too great, the plate will be crushed as is indicated at m. In like manner, if the equal and opposite pressure exerted by the plate upon the rivet is too great, the rivet will be crushed. Hence sufficient bearing surface must be provided between rivet and plate to avoid injury to either.

The mode of distribution of the pressure between rivet and plate is not well understood. It is, however, common practice to assume that the intensity of this pressure is the same as that of a square pin of side d in a square hole (Fig. 87), and on this assumption to determine experimentally the safe unit bearing stress between rivet and plate. Accordingly, *the bearing area between a rivet of diameter d in a plate of thickness t is taken as* $A_c = td$. Hence, if S_c is the safe unit bearing stress between rivet and plate,

$$P_c = A_c S_c = tdS_c$$

is the *safe strength of the joint in bearing.** If there are n rivets, each of diameter d, then $A_c = ntd$, and

$$P_c = ntdS_c.$$

c. Tension on Net Section.—The portion of the plate now to be considered is shown in Fig. 88. The entire joint is in tension.

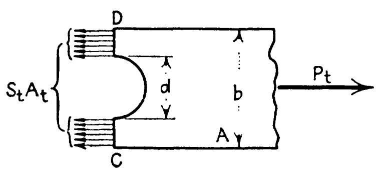


FIG. 88.

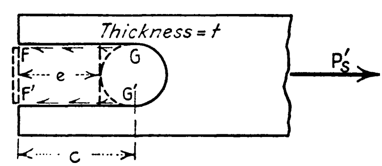


FIG. 89.

Evidently, the greatest unit tensile stress in the plate occurs on the section CD. Hence, if A_t is the net sectional area of the plate at the rivet hole, and S_t is the safe unit tensile stress (assumed to be uniformly distributed over the net section CD),

$$P_t = A_t S_t = (b - d)tS_t$$

is the *safe strength of the joint in tension.*

If there are m rivets in one row, each of diameter d, then for that row

$$A_t = (b - md)t \qquad \text{and} \qquad P_t = (b - md)tS_t.$$

d. Shear in Plate between Rivet Hole and End of Plate.—If the shearing resistance on the two rectangular areas FG and $F'G'$ (Fig. 89) has a combined value less than the force exerted by the rivet, the rivet may push the portion $F'G$ of the plate out toward

* If the intensity of pressure between rivet and plate were uniformly distributed in a radial direction [Art. 71, Eq. (c)], this expression for P_c would be theoretically correct.

the end of the plate. If A'_s is the combined area of the two rectangular surfaces FG and $F'G'$, and S_s is the safe unit shearing stress in the plate, then

$$P'_s = A'_s S_s$$

is the safe strength of the joint as far as shear in plate is concerned.

75. Note.—If the end of the plate is a sheared surface (cut by a shearing machine), it will be slightly damaged. Moreover, the plate may not start to shear at G and G' but nearer the end of the plate, as is indicated in Fig. 89. To be on the side of safety, assume that e (see figure) is the length of a shearing surface. Then the total shearing area is $A'_s = 2\ et$ and the total strength of the plate in shear is $P'_s = 2S_s\, et$ where t is the thickness of the plate. Hence the joint will be at least as strong in shear in the plate as it is in bearing between plate and rivet if

$$2S_s et = S_c td. \qquad (a)$$

Now the ultimate shearing stress in the plate is between 45,000 and 50,000 lb./sq. in. and the ultimate bearing stress between plate and rivet is between 90,000 and 95,000 lb./sq. in. That is, approximately,

$$S_c = 2S_s.$$

Substituting in Eq. (*a*),

$$2S_s et = 2S_s td,$$

or

$$e = d.$$

Or the center of the rivet hole is $d + d/2 = 1.5d$ from the end of the plate.

In this chapter, it will be assumed that *the center of the rivet hole is at least* 1.5 *diameters of the hole from the end of the plate and that therefore it will not be necessary to investigate shear in the plate in front of the rivet hole.* For instance, if the rivet hole is 1 in. in diameter, it will be assumed that the center of the rivet hole is at least 1.5 in. from the end of the plate.

If it is assumed that the center of the rivet hole is *at least* 1.5d from the end of the plate, there remain three main force actions that must be examined in the analysis of a riveted joint. They are

1. Shear in the rivet.
2. Bearing between rivet and plate.
3. Tension on a net section of a plate.

It should be noted that in the analysis of the riveted joint of Art. 74 it was assumed that the stress (shear, bearing, or tension) was uniformly distributed over the area on which it acts. It is not clearly understood just how a stress in a riveted joint is distributed. It is customary, however, to assume "uniform distribution" of stress and on this basis to *determine experimentally the safe or allowable unit stress.*

Example I.—In a lap joint with one rivet (Fig. 84), $t = \frac{3}{8}$ in., $b = 2$ in., $d = \frac{7}{8}$ in. Find the strength of this joint if $S_s = 10{,}000$ lb./sq. in., $S_c = 20{,}000$ lb./sq. in., and $S_t = 16{,}000$ lb./sq. in.
Strength in rivet shear:

$$P_s = A_sS_s = \tfrac{1}{4}\pi(\tfrac{7}{8})^2 \times 10{,}000 = 6013 \text{ lb.}$$

Strength in bearing:

$$P_c = A_cS_c = tdS_c = \tfrac{3}{8} \times \tfrac{7}{8} \times 20{,}000 = 6560 \text{ lb.}$$

Strength in tension:

$$P_t = A_tS_t = (2 - 0.875) \times \tfrac{3}{8} \times 16{,}000 = 6750 \text{ lb.}$$

If it is assumed that the plates are safe against shearing out in front of the rivet, it follows that the strength of the joint is

$$P = P_s = 6013 \text{ lb.}$$

Example II.—Let $t = \frac{3}{8}$ in., $b = 2.5$ in., $d = 1$ in. Find the strength of the joint if $S_s = 12{,}000$ lb./sq. in., $S_c = 24{,}000$ lb./sq. in., $S_t = 18{,}000$ lb./sq. in.

Ans. $P_s = 9425$ lb.; $P_c = 9000$ lb.; $P_t = 10{,}120$ lb.

Example III. Fig. 90.—Two $\frac{3}{8}$-in. plates, 13.25 in. wide, are joined together by means of a lap joint. There are nine $\frac{7}{8}$-in. rivets placed in three rows as shown (chain arrangement). Find the strength of the joint. Given $S_s = 13{,}500$ lb./sq. in., $S_c =$

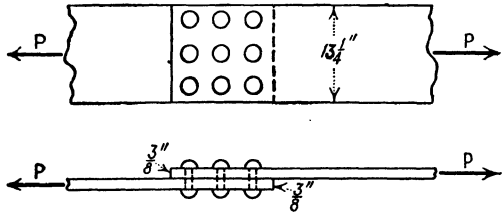


FIG. 90.

30,000 lb./sq. in., and $S_t = 18{,}000$ lb./sq. in. There are nine rivets in the joint. Each rivet is assumed to carry one-ninth of

the load. That is, the total strength of the nine rivets is assumed to be nine times that of one rivet.

Shear:

$$P_s = A_sS_s = 9 \times \frac{\pi}{4}\left(\frac{7}{8}\right)^2 \times 13{,}500 = 73{,}060 \text{ lb.}$$

Bearing:

$$P_c = A_cS_c = 9 \times \tfrac{3}{8} \times \tfrac{7}{8} \times 30{,}000 = 88{,}560 \text{ lb.}$$

Since there are three rivets in each row, the net section of the plate is the same for each row. Hence only the net section through the first row of rivets needs investigation to find the strength of the joint in tension. If the second row contained more rivets than the first row, it would be necessary (as will be seen later) to investigate also the net section through the second row of rivets.

Tension:

$$P_t = A_tS_t = (13.25 - 3 \times \tfrac{7}{8}) \times \tfrac{3}{8} \times 18{,}000 = 71{,}650 \text{ lb.}$$

Tension governs:

Therefore

$$P = P_t = 71{,}650 \text{ lb.}$$

76. The efficiency of a riveted joint is the ratio of the strength of the joint to the strength of the solid main plate. By "solid plate" is meant a plate without any rivet holes through it. Efficiency is usually expressed as a percentage. Since rivet holes reduce the effective sectional area of a plate, the efficiency of a riveted joint is always less than 100 per cent.

Example.—With reference to the joint of Example III of Art. 75 (Fig. 90), it was found that the safe strength of the joint is 71,650 lb. The safe strength of the solid plate is

$$13.25 \times \tfrac{3}{8} \times 18{,}000 = 89{,}500 \text{ lb.}$$

Therefore

$$\text{Efficiency} = \frac{71{,}650}{89{,}500} = 0.80 = 80 \text{ per cent.}$$

77. Single-strap Butt Joint.—Figure 91 represents a joint made by placing two (main) plates end to end and joining them by means of a single strap or cover plate. Such a joint is called a *single-strap butt joint* or a *single covered butt joint.* Figure 92 shows the right half of the joint taken free. Evidently, each

half of the joint is equivalent to a lap joint and therefore is analyzed in the same way as a lap joint.

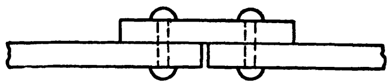


FIG. 91.

A single-strap butt joint has no advantage over the corresponding lap joint except that in the butt joint a fairly smooth surface may be obtained by countersinking the rivet heads on the side

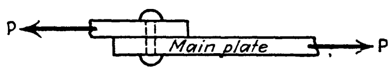


FIG. 92.

opposite that on which the strap is placed. The butt joint requires twice as many rivets as the lap joint and therefore increases the expense of punching holes and setting rivets.

78. Rivets in Double Shear.—Consider now a *double-strap butt joint* with one rivet through each main plate (Fig. 93).

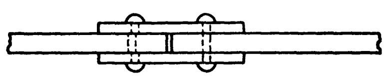


FIG. 93.

If failure of this joint is due to shearing of a rivet, the rivet must shear along two surfaces, as is indicated on the left side in the figure. Such a rivet is said to be in *double shear*. A rivet is twice as strong in double shear as in single shear. The total shearing area in a cylindrical rivet in double shear is, therefore,

$$A = \frac{2\pi d^2}{4}.$$

Rivets, bolts, and pins occasionally may be subjected to more than double shear. For example, in the chain hinge shown in Fig. 94 a pin is in quadruple shear.

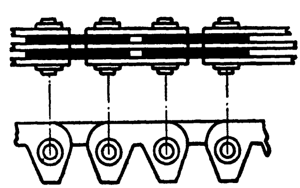


FIG. 94.

The nature of shearing action on rivets and pins in double shear is shown in Fig. 95, which gives sketches of laboratory specimens after the application of shearing forces such as are applied in riveted joints.

79. Double-strap Butt Joint.—The computation of the strength of a given riveted joint will now be made for the double-strap butt joint of Fig. 96.

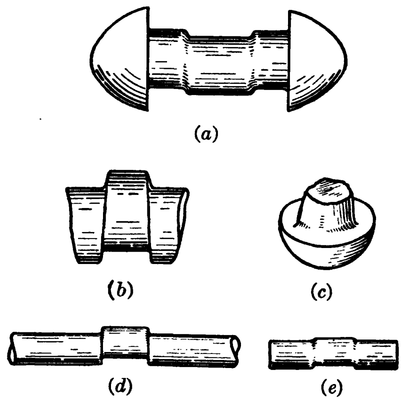


FIG. 95.—Shearing of (a) rivet in double shear; (b) headless rivet showing double shear; (c) failure of rivet in shear; (d) lead cylinder; (e) wooden pin.

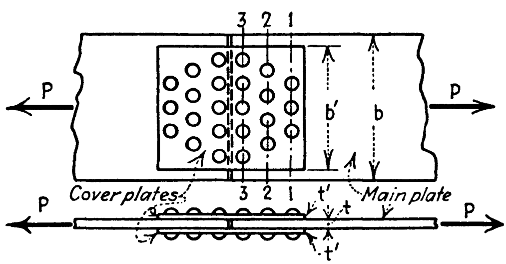


FIG. 96.

Let S_s = safe unit shearing stress in the rivets.
S_c = safe unit bearing stress between rivets and plates.
S_t = safe unit tensile stress in plates.
b = gross width of main plates.
b' = gross width of cover plates.
t = thickness of main plates.
t' = thickness of each cover plate.
d = diameter of driven rivet.
n = total number of rivets through each main plate,
= 12 in the given joint (Fig. 96).
m_1, m_2, m_3 = number of rivets in first, second, and third row, respectively. With reference to the joint of Fig. 96, $m_1 = 3$, $m_2 = 4$, and $m_3 = 5$.

Note.—The exact nature of the distribution of stress to the several rivets is not known. It is customary to assume that the load is divided equally among them. With reference to the joint of Fig. 96, it is assumed that each rivet carries one-twelfth of the load [see Art. 48 (2)].

Strength of Joint in Shear.—Since there are 12 rivets through each main plate ($n = 12$), each rivet being in double shear, the total shearing area is $A_s = 12 \times 2 \times \pi d^2/4$. Hence the strength of the joint in shear is

$$P_s = A_s S_s = 12 \times 2 \times \frac{\pi d^2}{4} S_s. \qquad (a)$$

Strength of Joint in Bearing. Main Plate.—The total bearing area between the plate and the 12 rivets is $A_c = 12td$. Hence the strength of the joint in bearing is

$$P_c = A_c S_c = 12tdS_c. \qquad (b)$$

Strength of Joint in Tension. Main Plate. First Row of Rivets. (1 . . . 1 in Fig. 96).—The net section of the main plate through the first row of rivets must carry the entire load P in tension. Hence, if P_t' designates the strength of the joint based upon the tensile strength of the main plate at the first row of rivets,

$$P_t' = A_t' S_t = (b - m_1 d)tS_t = (b - 3d)tS_t \qquad (c)$$

where A_t' = net section of the main plate at the first row of rivets.

Second Row of Rivets.—There are 12 rivets transmitting the entire load and each rivet is assumed to carry one-twelfth of the load. The first row of rivets has already transmitted three-twelfths of P before the second row is reached. This leaves nine-twelfths of P to be passed through the net area of section 2 . . . 2. Hence P must not be greater than that obtained from the equation

$$\frac{9P}{12} = A_t'' S_t = (b - m_2 d)tS_t = (b - 4d)tS_t$$

in which A_t'' = net sectional area of main plate at the second row of rivets.

Or, solving for P and designating its value by P_t'',

$$P_t'' = \tfrac{12}{9}(b - 4d)tS_t. \qquad (d)$$

This is the strength of the joint based upon the tensile strength of the main plate at the second row of rivets.

Third Row of Rivets.—The tension in the main plate through the third row of rivets is $5P/12$. Hence P must not be greater than that obtained from the equation

$$\tfrac{5}{12}P = A_t'''S_t = (b - 5d)tS_t.$$

That is, the strength of the joint based upon the tensile strength of the main plate at the third row of rivets is

$$P_t''' = \tfrac{12}{5}(b - 5d)tS_t. \qquad (e)$$

Bearing in Cover Plates.—If the combined thickness of the two cover plates equals the thickness of the main plate, there will be as much bearing area in the cover plates as there is in the main plate. Hence, if $t' = t/2$ or $t' > t/2$, bearing in the cover plates need not be investigated. For various reasons, it is the practice to select cover plates having a combined thickness a little greater than the thickness of the main plate.

Tension in Net Area of Cover Plates.—Take free the right half of the joint (Fig. 97). For equilibrium to exist, the tension

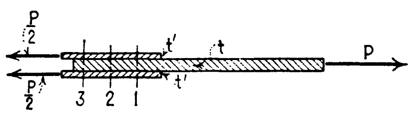


FIG. 97.

in each cover plate is one-half of P as shown. Evidently, the maximum tensile stress in a cover plate occurs on a section through the third row of rivets, 3 . . . 3. Hence P must not be greater than the value found from the equation

$$\frac{P'}{2} = (b' - 5d)t'S_t. \qquad (f)$$

The smallest of the values of P as determined by the several equations [Eqs. (*a*) to (*f*)] of this article gives the strength of the joint.

Example. Fig. 98.—In a double-strap butt joint, $b = 14.25$ in. $= b'$, $t = 0.75$ in., $t' = 0.50$ in., $d = 1$ in., and $n = 9$. The rivets are arranged as shown.
Find the strength of the joint; given $S_s = 12{,}000$ lb./sq. in., $S_c = 24{,}000$ lb./sq. in., and $S_t = 18{,}000$ lb./sq. in.
Shear:

$$P_s = 9 \times 2 \times \frac{\pi 1^2}{4} \times 12{,}000 = 169{,}700 \text{ lb.}$$

Bearing, main plate:

$$P_c = 9 \times 0.75 \times 1 \times 24{,}000 = 162{,}000 \text{ lb.}$$

Tension, main plate.

First row:

$$P'_t = (14.25 - 2) \times 0.75 \times 18{,}000 = 165{,}500 \text{ lb.}$$

Second row:

$$\frac{7P}{9} = (14.25 - 3) \times 0.75 \times 18{,}000 = 152{,}000$$

or

$$P''_t = 195{,}500 \text{ lb.}$$

Third row:

$$\tfrac{4}{9}P = (14.25 - 4) \times 0.75 \times 18{,}000 = 138{,}500$$

or

$$P'''_t = 312{,}000 \text{ lb.}$$

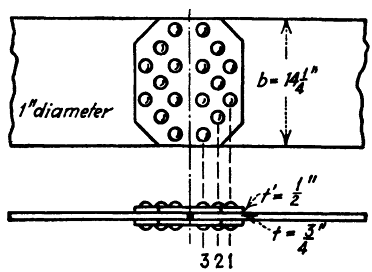


FIG. 98.

Tension, cover plates:

$$\frac{P'}{2} = (14.25 - 4) \times 0.5 \times 18{,}000,$$

or

$$P' = 184{,}500 \text{ lb.}$$

Bearing governs,

Therefore

$$P = P_c = 162{,}000 \text{ lb.}$$

$$\text{Efficiency} = \frac{162{,}000}{14.25 \times \frac{3}{4} \times 18{,}000} = 0.842 = 84.2 \text{ per cent.}$$

80. Rivets. Rivet Holes.—Riveting is done usually by machines (pressure machines in the shop and pneumatic hammers in the field). A rivet having been heated to a red heat is put into the rivet hole and the head is formed by pressing or hammering. As a result of the driving, the body of the rivet expands. For a good joint the driven rivet (finished rivet) should fill the hole completely.*

* Rivets sometimes are driven cold. Such rivets are stronger than rivets driven hot. A cold rivet, however, is much harder to drive than a hot rivet.

Since the heated rivet must be inserted into the rivet hole, the hole must be slightly larger than the nominal diameter of the rivet. By "nominal diameter of the rivet" is meant the original diameter, *i.e.*, the diameter of the rivet before it is heated and driven. In practice, as a rule, the rivet hole is made $\frac{1}{16}$ in. larger than the nominal diameter of the rivet. Thus, if a $\frac{7}{8}$-in. rivet (nominal size $\frac{7}{8}$ in.) is used, the hole is made $\frac{15}{16}$ in. in diameter ($\frac{7}{8} + \frac{1}{16} = \frac{15}{16}$).

When rivet holes are punched, the parts (plates, angles, channels, etc.) are punched separately. In the assembly of the parts the holes seldom match perfectly. For instance, in the joint of Fig. 98, 9 rivet holes must be punched through each main plate and 18 rivet holes through each cover plate. When the holes are punched, the plates are punched one at a time. On assembling the plates it seldom happens that the holes through the cover plates agree perfectly with the holes through the main plates; *i.e.*, the holes do not match or register perfectly. Moreover, the holes are not always perfect. It frequently happens, therefore, that the *effective* diameter of the driven rivet is less than the diameter of the rivet hole. In some cases, the effective diameter of the driven rivet is no greater than the nominal diameter of the rivet. In computing the strength of a rivet in shear and in bearing (holes punched to size), it is a common practice to use the nominal diameter of the rivet.

Rivet holes usually are made by punching the plates cold. Punching, however, injures the material immediately around a hole. It is necessary, therefore, to make allowance for the injured material around a punched hole when the tensile strength of the plate is determined. This is done, as a rule, by adding $\frac{1}{16}$ in. to the diameter of the punched hole when the net sectional area of the plates is determined. For instance, if holes are punched to size a rivet hole should be taken $\frac{1}{8}$ in. greater than the normal diameter of the rivet, *i.e.*, $\frac{1}{16}$ in. to allow for the damaged material around the hole, and another $\frac{1}{16}$ in., since the rivet hole is $\frac{1}{16}$ in. larger than the normal diameter of the rivet. Thus, if a $\frac{7}{8}$-in. rivet (normal size) is used, the rivet hole (if punched to size) should be taken as 1 in. ($\frac{7}{8} + \frac{1}{8}$) when determining the net width of the plate.

In high-class work, rivet holes are subpunched (punched undersize). The parts (plates, angles, etc.) are assembled and rivet holes are then reamed to size. As a result, the holes are

smooth and match. Moreover, reaming removes the damaged material around the hole.*

In this text it will be assumed that rivet holes are *reamed to size* and that (as specified in the Boiler Code) the *diameter of the driven rivet* and *the diameter of the reamed hole may be used* in determining the strength of a joint or designing a joint to carry a given load. Moreover, when the diameter of a rivet is specified, assume that this is the diameter of the driven rivet (which equals that of the reamed hole).

Riveted joints are important elements in structures and give practical illustrations of the simultaneous occurrence of tension, shear, and compression. The theory underlying riveted joints should be clearly understood. In this text no single code will be followed completely. If it is desired to comply fully with a particular code, the necessary modifications can readily be made.

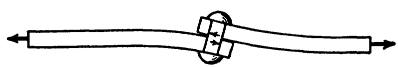


FIG. 99.

Note.—The consensus of the opinions of experts as to what shall constitute good practice in the design of a particular kind of construction is embodied in a "code" in the form of specifications. Some of these specifications are important "shop rules." Others are based partly on theoretical considerations and partly on practical considerations. A code is a protection to the public since it fixes a standard.

81. Unit Stresses.—The selection of the unit stresses for a joint depends in an important way upon the type of structure and its use. It also depends upon the way the rivet is driven. Field rivets usually are driven without being bucked by pneumatically or electrically driven machines. Such rivets are called *hand-driven rivets.* The conditions in the field are often difficult to control, and the codes specify lower unit stresses for hand-driven rivets.

In a lap joint (Fig. 99) the pulls in the two plates are not colinear. Hence there is a tendency for the plates to bend as is indicated in the figure (bending exaggerated). Owing to the

* In special cases (high-class marine work, for instance) the solid parts are assembled and rivet holes are then drilled.

bending of the plates the bearing between rivet and plate is not uniformly distributed. The code adopted by the American Institute of Steel Construction (A. I. S. C.) specifies a lower unit bearing stress for rivets in single shear. On the other hand, the Structural Code for Railway Bridges (also the Boiler Code) makes no distinction between rivets in single shear and rivets in double shear so far as unit bearing stress is concerned. In this text the *unit bearing stress for rivets in single shear will be taken the same as that for rivets in double shear.*

82. Shearing and Bearing Values of Rivets.—The strength of a cylindrical rivet is

In single shear,

$$R'_s = \frac{\pi d^2 S_s}{4}.$$

In double shear,

$$R''_s = 2R'_s.$$

and in bearing in a plate of thickness t,

$$R_c = S_c t d.$$

Values of R'_s, R''_s, and R_c may be calculated for different sizes of rivets, for different thicknesses of plates, and for different

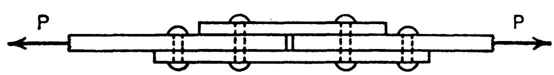


FIG. 100.

unit stresses. A table of such values is very convenient when the strength of a joint in shear or in bearing is determined.

If there are n rivets through each main plate, the strength of the joint is

In shear,

$P_s = nR'_s$ (lap joint, or single-strap butt joint),

or

$P_s = nR''_s = 2nR'_s$ (double-strap butt joint, all rivets being in double shear.*

In bearing,

$$P_c = nR_c.$$

* For a joint such as is shown in Fig. 100, if n_1 is the number of rivets in single shear and n_2 is the number in double shear, the total strength of the joint in shear is $P_s = n_1R'_s + n_2R''_s$.

TABLE I.—SHEARING AND BEARING VALUES OF RIVETS IN KIPS
Unit stresses;* S_s = 15,000 lb./sq. in.; S_c (s.s.) = 32,000 lb./sq. in.; S_c (d.s.) = 40,000 lb./sq. in.

d diameter of rivet, inches	*A* area of section, sq. in.	Shearing value, kips		R_c, bearing value, kips				
				t, thickness of plate, inches				
		R'_s Single shear	R''_s Double shear	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	
$\frac{1}{2}$	0.1963	2.95	5.89	3.00 3.75	..... 5.00	....	....	32.0 40.0
$\frac{5}{8}$	0.3068	4.60	9.20	3.75 4.69	..... 6.25	..... 9.38	....	32.0 40.0
$\frac{3}{4}$	0.4418	6.63	13.25	4.50 5.62	6.00 7.50	..... 11.3	....	32.0 40.0
$\frac{7}{8}$	0.6013	9.02	18.04	5.25 6.56	7.00 8.75	..... 13.1	.... 17.5	32.0 40.0
1	0.7854	11.78	23.56	....	8.00 10.00	12.0 15.0	.... 20.0	32.0 40.0
$1\frac{1}{8}$	0.9940	14.91	29.82	....	....	13.5 16.9	.... 22.5	32.0 40.0

* From A.I.S.C. Manual, 1941.

By referring to this table, it is seen that the shearing value of a $\frac{3}{4}$-in. rivet in single shear is R'_s = 6630 lb. and that its bearing value in a $\frac{1}{4}$-in. plate is R_c = 6000 lb. These values are obtained as follows:

$$R'_s = 15{,}000 \times \frac{\pi}{4} \times \left(\frac{3}{4}\right)^2 = 6630 \text{ lb.}$$

$$R_c = 32{,}000 \times \tfrac{3}{4} \times \tfrac{1}{4} = 6000 \text{ lb.}$$

TABLE II.—SHEARING AND BEARING VALUES OF RIVETS
Unit stresses; S_s = 10,000 lb./sq. in., S_c = 20,000 lb./sq. in.

d Diameter of rivet, inches	*A* Area of section, sq. in.	Shearing value, lb.		R_c, bearing value, lb.					
				t, thickness of plate, inches					
		R'_s Single shear	R''_s Double shear	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
$\frac{1}{2}$	0.1964	1,964	3,927	2,500	3,750	5,000			
$\frac{5}{8}$	0.3068	3,068	6,136	3,125	4,688	6,250			
$\frac{3}{4}$	0.4418	4,418	8,836	3,750	5,625	7,500	9,375		
$\frac{7}{8}$	0.6013	6,013	12,026	4,375	6,563	8,750	10,938	13,125	
1	0.7854	7,854	15,708	.....	7,500	10,000	12,500	15,000	
$1\frac{1}{8}$	0.9940	9,940	19,880	.....	.....	11,250	14,063	16,875	19,688

Example I.—Ten $\frac{7}{8}$-in. rivets are used in a lap joint (being therefore in single shear). The thickness of the plate is $t = \frac{1}{2}$ in., $S_s = 10{,}000$ lb./sq. in., $S_c = 20{,}000$ lb./sq. in.

From Table II, it is seen that $R'_s = 6013$ lb. and $R_c = 8750$ lb. Hence, so far as shear and bearing are concerned, shear governs. The strength of the joint in bearing need not be calculated, since the joint will be stronger in bearing than in shear.

$$P_s = 10 \times 6013 = 60{,}130 \text{ lb.}$$

Example II.—If the joint is a double-strap butt joint ($\frac{7}{8}$-in. rivets, $\frac{1}{2}$-in. main plates, 10 rivets), the rivets will be in double shear. From the table, $R''_s = 12{,}026$ lb. and $R_c = 8750$ lb. Hence, so far as shear and bearing are concerned, bearing now governs and shear need not be calculated.

$$P_c = 10 \times 8750 = 87{,}500 \text{ lb.}$$

Example III.—A joint is to sustain a load of $P = 30{,}000$ lb. The thickness of plates is $\frac{3}{8}$ in. and the diameter of rivets is $d = 1$ in.

If $S_s = 15{,}000$ lb./sq. in. and $S_c = 32{,}000$ lb./sq. in., how many rivets are required for a lap joint? For a double-strap butt joint?

Lap Joint.—For one rivet, $R'_s = 11{,}780$ lb. and $R_c = 12{,}000$ lb. (see Table I). Hence shear governs and the required number of rivets is $30{,}000/11{,}780 = 2.5$. Therefore $n = 3$.

Double-strap Butt Joint: $R''_s = 23{,}560$ lb. and $R_c = 15{,}000$ lb. Bearing governs and the number of rivets required is

$$\frac{30{,}000}{15{,}000} = 2. \qquad \text{Therefore } n = 2.$$

PROBLEMS

44. In a lap joint, $b = 10\frac{1}{2}$ in., $t = \frac{1}{2}$ in., $d = \frac{7}{8}$ in., and $n = 9$. Rivets are arranged in three rows of three rivets in a row. Required to find the strength and the efficiency of the joint.

Given $S_s = 10{,}000$ lb./sq. in., $S_c = 20{,}000$ lb./sq. in., $S_t = 16{,}000$ lb./sq. in. *Ans.* $P = 54{,}117$ lb.; efficiency = 64.4 per cent.

45. In Problem 44, a double-butt strap joint is used. Assume cover plates $\frac{5}{16}$ in. thick. *Ans.* $P = 63{,}000$ lb.; efficiency = 75 per cent.

46. In Problem 45, arrange rivets as follows: two in the first row, three in the second, and four in the third (see Art. 79).

Ans. $P = 70{,}000$ lb.; efficiency = 83.3 per cent.

Note.—Compare the results of the three problems and note that the efficiency of the joint depends not only upon the arrangement of rivets but also upon the type of joint used.

83. Riveted Seams. Continuous Joints.—The joints so far considered have been *structural joints, i.e.,* joints used in buildings, bridges, and other structures. There is another class of joints called *continuous joints.* These joints occur, for instance, where bent plates are riveted (or welded) together to form a boiler, a tank, or a large water pipe (Fig. 101). A riveted continuous joint often is called a *riveted seam.* The continuous joints considered here will be assumed to be riveted.

In dealing with a riveted seam, it is convenient to limit considerations to a minimum width of plate containing a typical or

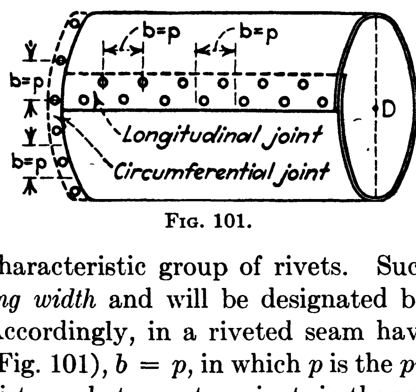


Fig. 101.

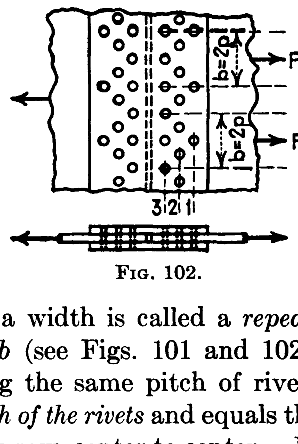


Fig. 102.

characteristic group of rivets. Such a width is called a *repeating width* and will be designated by b (see Figs. 101 and 102). Accordingly, in a riveted seam having the same pitch of rivets (Fig. 101), $b = p$, in which p is the *pitch of the rivets* and equals the distance between two rivets in the same row, center to center. In a seam of different pitch, b equals the maximum pitch. Thus (Fig. 102) $b = 2p$ where p equals the minimum pitch. It should be noted that a repeating width may be viewed in a number of ways (see figures). The number of rivets in a repeating width will be designated by n. In the circumferential joint of Fig. 101, $n = 1$; in the longitudinal joint, $n = 2$; in the joint of Fig. 102, $n = 5$.

A joint is said to be *single riveted, double riveted, triple riveted,* etc., according as the rivets through each main plate are arranged in one, two, three, etc. rows. Thus (Fig. 101) the circumferential joint is a *single-riveted lap joint,* and the longitudinal joint is a *double-riveted lap joint of the same pitch* (two rows of rivets, pitch the same for both rows). Figure 102 shows a *triple-riveted double-*

butt strap joint of different pitch (three rows of rivets, two cover plates, pitch not the same for the three rows through each main plate).

84. Investigation of a Riveted Seam.—As a rule, the investigation of the strength of a given riveted seam is a simple matter and is similar to that for a structural joint.

Example. Fig. 102.—In a continuous triple-riveted double-strap butt joint of different rivet pitch, the minimum pitch is $p = 3\frac{5}{8}$ in. Thickness of main plates is $t = \frac{3}{4}$ in. Diameter of rivet is $d = 1$ in. Assuming that the combined strength of the cover plates is at least equal to the strength of a main plate, find the strength of the joint per linear inch of seam. Also find the efficiency of the joint. Note that there are five rivets in a repeating width and that the first row contains one rivet. Given, then, $n = 5$, $b = 2p = 7.25$ in., $d = 1$ in., and

$$t = 0.75 \text{ in.}$$

Take $S_s = 10{,}000$ lb./sq. in., $S_c = 20{,}000$ lb./sq. in., and $S_t = 16{,}000$ lb./sq. in.

From Table II (Art. 82) $R_s'' = 15{,}708$ lb., $R_c = 15{,}000$ lb. Bearing governs:

$$P_c = nR_c = 5 \times 15{,}000 = 75{,}000 \text{ lb.}$$

Tension.

First row:

$$P_t' = (7.25 - 1) \times 0.75 \times 16{,}000 = 75{,}000 \text{ lb.}$$

Second row:

$$\tfrac{4}{5}P = (7.25 - 2) \times 0.75 \times 16{,}000$$

or

$$P_t'' = 78{,}750 \text{ lb.}$$

Therefore the strength of the joint for the repeating width is $P = 75{,}000$ lb. The strength per linear inch of seam is

$$T = \frac{75{,}000}{7.25} = 10{,}340 \text{ lb./lin. in.}$$

Efficiency:

$$\eta = \frac{10{,}340}{1 \times 0.75 \times 16{,}000} \times 100 = 86.2 \text{ per cent.}$$

Problem 47. Fig. 101.—In a tank, the longitudinal seam is a double-riveted lap joint of the same pitch. Given $t = \frac{5}{16}$ in., $d = \frac{3}{4}$ in., $p = 2\frac{1}{2}$ in.,

S_s = 10,000 lb./sq. in., S_c = 20,000 lb./sq. in., and S_t = 16,000 lb./sq. in. Find the strength of the joint per linear inch of seam and also the working efficiency.

Ans. P_s = 8840 lb., P_t = 8750 lb. Therefore P = 8750 lb.
T = 3500 lb./lin. in. of seam, and efficiency = 70.0 per cent.

85. Design of Riveted Joints. *Riveted Seams.*—It is important to keep the efficiency of a riveted seam as high as practicable. Suppose that a long water pipe is to be installed. If the efficiency of the seam in the pipe is 60 per cent, then 40 per cent of the metal in the pipe between seams is not needed for strength. If the efficiency is 85 per cent, only 15 per cent of the metal between seams is not needed. It can be shown that the second pipe requires about 30 per cent less metal in the main plates than the first pipe. In general, increasing the efficiency of the riveted seams in a pipe (or boiler, tank, etc.) tends to decrease the cost of the pipe.

A joint of a given type is called a joint of *maximum efficiency* if it is of equal strength in shear, bearing, and tension, *i.e.*, if it is a *balanced joint.* Such a joint is an ideal joint but it is seldom practicable. It is apt to require too large a rivet, making difficulty in driving, or the pitch may be too large. If the pitch is too large, the seam cannot be made watertight. Moreover, commercially available sizes of plates and rivets must be used. Owing to such practical limitations upon the design of a riveted seam, it is seldom possible to make P_s, P_c, and P_t equal to each other.

Standard designs of riveted seams have been made so as to attain as high an efficiency as the combination of theoretical and practical considerations will permit. These designs are taken as a guide in designing riveted seams. The design of riveted seams is beyond the scope of this text.

Structural Joints.—As a rule, much greater freedom is given in the design of a structural joint than in the design of a riveted seam. Frequently, however, a structural joint is subjected to conditions that complicate the design. The joint may, for instance, be subjected to a bending moment. In all cases it is necessary to comply with shop rules, etc. The Structural Code specifies that *the minimum distance between centers of rivet holes shall be three diameters of the rivet.* If rivets are placed too closely together, the holes cannot be conveniently made or the heads of the rivets formed. In conformity with the rule of

spacing, if b is the gross width of plate and m is the number of rivets in a row, then the minimum gross width of the plate through that row of rivets is

$$b_{\text{min.}} = 3md.$$

In Fig. 103, if $d = \frac{3}{4}$ in. and $m = 2$, then

$$\text{min. } b = 3 \times 2 \times \tfrac{3}{4} = 4.5 \text{ in.}$$

Hence the plate must be at least 4.5 in. wide; it may, of course, be much wider.

A detailed treatment of the design of riveted structural joints is beyond the scope of this text. It seems sufficient to give two simple but typical examples.

Example I.—Two $\frac{3}{8}$-in. plates are connected by a lap joint. The joint is to carry safely a load of $P = 80{,}000$ lb. Design

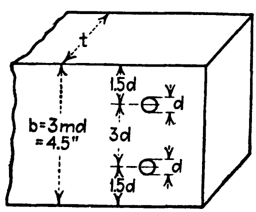


FIG. 103.

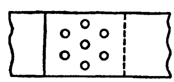


FIG. 104.

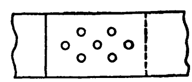


FIG. 105.

the joint using 1-in. rivets. Given $S_s = 15{,}000$ lb./sq. in., $S_c = 32{,}000$ lb./sq. in., and $S_t = 20{,}000$ lb./sq. in. Put not more than three rivets in a row. (Note that the problem has already been simplified by the data and the assumptions.)

1. *Thickness of Plates.*—Given $t = \frac{3}{8}$ in.

2. *Required Number of Rivets.*—From Table I (Art. 82), for 1-in. rivets and $\frac{3}{8}$-in. plate, $R_s = 11{,}780$ lb. while $R_c = 12{,}000$ lb. Hence shear governs, being the lesser value. (It is clear that the strength is already unbalanced.) The number of rivets required is

$$n = \frac{80{,}000}{11{,}780} = 6.8.$$

Of course, seven rivets must be used. Take $n = 7$.

3. *Arrangement of Rivets.*—Several layouts of the rivet group are possible. Assume the rivets to be arranged as shown in Fig. 104.

4. *Required Gross Width for Tension.*

First row:

$$80{,}000 = (b_1 - 2 \times 1) \times 0.375 \times 20{,}000.$$

Therefore

$$b_1 = 12\tfrac{2}{3} \text{ in.}$$

Second row:

$$\tfrac{5}{7} \times 80{,}000 = (b_2 - 3 \times 1) \times 0.375 \times 20{,}000.$$

Therefore

$$b_2 = 10.62 \text{ in.}$$

5. *Rule of spacing* demands that

$$b_{\text{min.}} = 3md = 3 \times 3 \times 1 = 9 \text{ in.}$$

6. Nearest commercial size above $12\frac{2}{3}$ in. is 13 in. Therefore

$$b = 13 \text{ in.}$$

Hence take plates 13 in. wide with rivets arranged as shown in Fig. 104.

7. *Efficiency:**

$$\eta = \frac{80{,}000}{13 \times 0.375 \times 20{,}000} \times 100 = 82 \text{ per cent.}$$

Example II.—In the joint of Example I, arrange the rivets as shown in Fig. 105.

Ans. $b_1 = 11\frac{2}{3}$ in.; $b_2 = 11.16$ in.; $b_{\text{min.}} = 6$ in.

Therefore

$$b = 11\tfrac{3}{4} \text{ in.}; \ \eta = 90.7 \text{ per cent.}$$

Example III.—A double-strap butt joint is to carry a load of $P = 120{,}000$ lb. Design the joint using $\frac{7}{8}$-in. rivets. $S_s = 10{,}000$ lb./sq. in., $S_c = 20{,}000$ lb./sq. in., $S_t = 16{,}000$ lb./sq. in. Put not more than five rivets in a row.

1. *Thickness of Main Plates.*—Rivets are in double shear. $R_s'' = 12{,}026$ lb. (Table II, Art. 82). Note that if a $\frac{3}{4}$-in. plate is used, $R_c = 13{,}125$ lb. That is, since a commercial thickness of plate must be used a $\frac{3}{4}$-in. plate makes the strength of a $\frac{7}{8}$-in. rivet in bearing as nearly equal as possible to the strength of this rivet in shear. Therefore take $t = \frac{3}{4}$ in.

* Since commercial sizes of plates must be used, it frequently happens that the strength of the joint is slightly greater than the load for which the joint is designed. It is customary, however, to base the efficiency of a joint on the load for which the joint is designed.

2. *Required Number of Rivets.*—Shear governs with

$$R_s'' = 12{,}026 \text{ lb.}$$

$$n = \frac{120{,}000}{12{,}026} = 10 \text{ (nearly). Use 10 rivets.}$$

3. *Arrangement of Rivets.*—Several designs are possible. Assume the rivets arranged as shown in Fig. 106.

4. *Required gross width* for tension. Main plates.

First row:

$$120{,}000 = (b_1 - \tfrac{7}{8}) \times \tfrac{3}{4} \times 16{,}000.$$
$$b_1 = 10\tfrac{7}{8} \text{ in.}$$

Second row:

$$\tfrac{9}{10} \times 120{,}000 = (b_2 - 2 \times \tfrac{7}{8}) \times \tfrac{3}{4} \times 16{,}000.$$
$$b_2 = 10\tfrac{3}{4} \text{ in.}$$

Since $b_2 < b_1$, it is not necessary to test the third row.

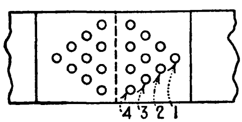


FIG. 106.

5. Rule of spacing requires that

$$b_{\text{min.}} = 3md = 3 \times 4 \times \tfrac{7}{8} = 10.5 \text{ in.}$$

6. Take $b = 11$ in. with rivets arranged as shown in Fig. 106.

7. *Efficiency:*

$$\eta = \frac{120{,}000}{11 \times 0.75 \times 16{,}000} \times 100 = 90.9 \text{ per cent.}$$

8. *Cover plates:*

Required thickness of cover plates through row 4. Assume cover plates to have the same width as main plates.

$$\left(11 - 4 \times \frac{7}{8}\right) \times t' \times 16{,}000 = \frac{P}{2} = 60{,}000.$$
$$t' = 0.50 \text{ in.} \qquad \text{Take } t' = \tfrac{1}{2} \text{ in.}$$

Example IV.—In the joint of Example III, put the rivets in two rows, with five rivets in a row (chain arrangement).

$$b = 14\tfrac{3}{8} \text{ in.}, b_{\text{min.}} = 13\tfrac{1}{8} \text{ in.}$$

Therefore

$$b = 14\tfrac{1}{2} \text{ in.}$$

and the efficiency is $\eta = 68.9$ per cent.

PROBLEMS

48. Fig. 107.—A double-riveted continuous lap joint is made of $\frac{3}{8}$-in. plates of steel. If $\frac{7}{8}$-in. rivets are used, what spacing of rivets will give maxi-

mum efficiency? Let S_s = 12,000 lb./sq. in., S_c = 24,000 lb./sq. in., and S_t = 18,000 lb./sq. in.

Suggestion.—For maximum efficiency, the strength in tension must equal the strength in shear or in bearing, whichever one governs.

Ans. x = 3.01 in.

49. A lap joint (with rivets hand driven) is to carry 54,000 lb. Three-eighths-in. plates and $\frac{7}{8}$-in. rivets are to be used. Design the joint.

50. A double-strap butt joint is to carry a load of 124,000 lb. (with rivets power driven). Using $\frac{7}{8}$-in. rivets, design the joint.

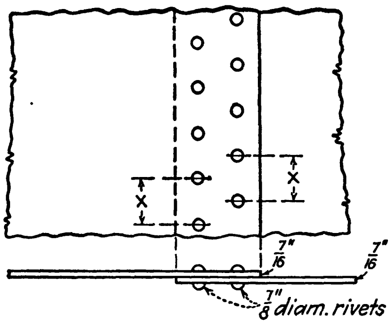


FIG. 107.

WELDED JOINTS

86. Introduction.—The theory of welded joints is not the same as that of riveted joints. Since welded joints are coming more and more into use, a brief discussion of their features, advantages, and the simpler methods used in their design will be included in this chapter.

A welded joint is made, as a rule, by allowing fused metal to flow between the parts or pieces that are to be joined together. The methods of welding in use at present are

1. *Gas Welding.*—That is, welding by the use of the oxyacetylene, oxyhydrogen, or some other flame to fuse the metal.

2. *Arc welding,* using the electric arc.

3. *Thermit Welding.*—This process for welding iron and steel is dependent upon the strong chemical affinity between powdered aluminum and powdered iron oxide. The fine powders are mixed in the most suitable proportions and the chemical process is started by igniting the mixture. The temperature produced is stated to be from 5000 to 5400°F. This melts the iron to a white heat, and the molten mass of iron occupies the lower part of the form or container, while the aluminum oxide appears as a floating slag. The chemical reaction is given by the equation

$$2Al + Fe_2O_3 = Al_2O_3 + 2Fe.$$

Arc welding and gas welding are the methods most commonly used, the weld metal consisting of a slender rod of the same material as the parts or pieces to be joined by the weld. Some-

times welding is done without the use of weld rods, the two pieces being fused together usually by means of an electric arc.

FIG. 108.

The various terms used in welding are as follows:

Butt Weld.—In a butt weld, the two main plates are butted squarely together (not necessarily in contact) and the weld metal is fused with the metal of both main plates. The edges of the main plates usually are beveled as shown in Fig. 108.

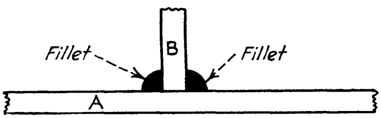


FIG. 109.

Fillet Weld.—A fillet weld is made at the intersection of two surfaces approximately perpendicular to each other. In Fig. 109, the two plates A and B are joined together by means of two fillets, one on each side of B. A fillet weld has a section approximately triangular in shape, the two narrow surfaces (corresponding to the legs of the right triangle) being fused with the surfaces of the pieces to be joined, the third surface (corresponding to the hypotenuse of the right triangle) being exposed to the air. Figure 110 shows a fillet on an enlarged scale. Figure 111 shows a welded lap joint. Note that the two plates are joined together by two fillets (shaded in the figure). The weld of Fig. 111 is a continuous weld. A *continuous weld* differs from an *intermittent weld* in having no alternating unwelded spaces.

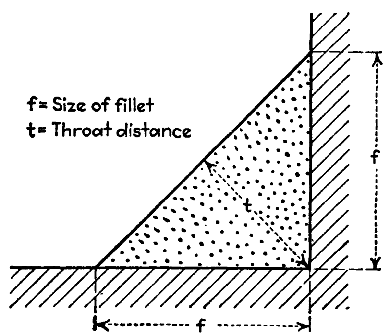


FIG. 110.

Tack Weld.—A tack weld is a short weld used to hold pieces together in assembling.

87. Design of Fillet Weld.—In structural joints made by welding, the fillet weld is generally used. From the results of a large number of tests on actual welded structural joints, the load per linear inch of fillet is usually specified as follows:

Size of fillet (see Fig. 110), inches....	$\frac{1}{2}$	$\frac{7}{16}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{1}{4}$
Load per linear inch of fillet, pounds...	4000	3500	3000	2500	2000

These are safe or working values.

Example I.—Referring to Fig. 111, let each fillet be 8 in. in length ($L = 8$ in.), and the size of the fillet be $\frac{3}{8}$ in. ($f = \frac{3}{8}$ in., see

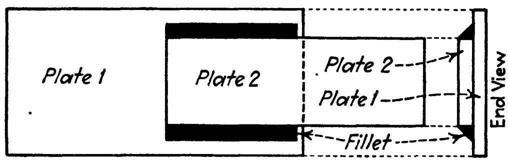


FIG. 111.

Fig. 110). Since there are two fillets, the total length of fillet is $2L = 16$ in. Therefore the total load which the joint can carry is

$$P = 16 \times 3000 = 48{,}000 \text{ lb.}$$

Example II.—A structural butt joint is made by welding the cover plates to the main plates (Fig. 112). Note that this is not a butt weld (Fig. 108) but a butt joint with fillet welds.

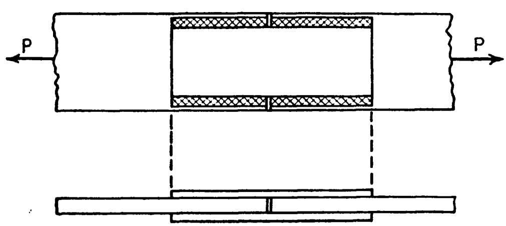


FIG. 112.

The main plates are 12 by $\frac{3}{4}$ in. and the cover plates are 10 in. wide. With an allowable unit tensile stress of

$$S_t = 16{,}000 \text{ lb./sq. in.,}$$

design the joint for 100 per cent efficiency.

Thickness of Cover Plates.—Each cover plate must carry one-half of the total load in the main plate.

Therefore

$$10 \times t \times 16{,}000 = \frac{12 \times \frac{3}{4} \times 16{,}000}{2},$$

or

$$t = 0.45 \text{ in.}$$

Take $t = 0.50$ in.

For a $\frac{1}{2}$-in. fillet, the strength of the fillet per linear inch equals 4000 lb. If L is the length of a fillet, there being four fillets, $4L \times 4000 = 12 \times \frac{3}{4} \times 16{,}000$, or $L = 9$ in.

Example III.—Design a welded connection for a $3\frac{1}{2}$- by $2\frac{1}{2}$- by $\frac{5}{16}$-in. angle and a gusset plate to develop the full tensile strength of the angle at 16,000 lb./sq. in.

$$P = S_tA = 16{,}000 \times 1.78 = 28{,}480 \text{ lb.}$$

Assume a $\frac{1}{4}$-in. fillet at 2000 lb./linear in.

$$\text{Length of weld} = \frac{28{,}480}{2000} = 14.24 \text{ in.} = a + b \qquad (1)$$

The tension P in the angle is considered acting along the gravity axis of the angle. So the lengths of the side welds are inversely proportional to their respective distances from the gravity axis of the angle.

$$\frac{b}{a} = \frac{1.14}{2.36} \qquad \text{or} \qquad b = 0.483a. \qquad (2)$$

Substituting the value of b, from Eq. (2), in Eq. (1),

$$a + 0.483a = 14.24$$
$$a = 9.6 \text{ in.}$$

and

$$b = 0.483 \times 9.6 = 4.64 \text{ in.}$$

88. Advantages of Welded Joints.—In general, it may be said that welding avoids the main disadvantages of riveted, bolted,

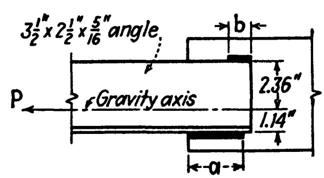


FIG. 113.

and keyed joints. In the first place, no holes, grooves, or openings need to be made in the main plates which are to be joined. Second, initial stresses due to forcing of parts into place are avoided. Third, the welded joint does not rely upon friction to develop any part of the strength of the joint.

Welded structures require less metal in a structure. Welding is adapted to joints in large-sized pieces and to parts of irregular shape. The operation of welding is relatively noiseless compared with the annoying clamor of riveting. Welding may be used with great success in making repairs to broken parts and also to add reinforcing pieces to parts of a structure which may have been weakened locally. Welded joints may be perfectly watertight and are less subject to corrosion because of the reduced amount of exposed area.

PROBLEMS

In the following problems, unless a statement to the contrary is made, results are given on the assumption that the Boiler Code of the American Society of Mechanical Engineers is used, *i.e.*, on the stipulation that rivet

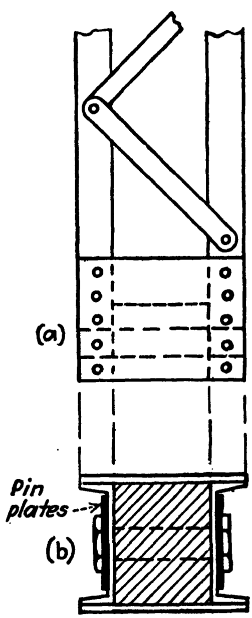


FIG. 114.

holes are reamed to size, and that the diameter of the driven rivet is used in calculating the strength of a rivet in shear and in bearing. The given rivet diameter is assumed to be that of the driven rivet.

51. Two $\frac{3}{8}$-in. plates, 14 in. wide, are fastened together by means of a lap joint. Twelve rivets, $\frac{3}{4}$ in. in diameter, are used, arranged in 3 rows having 4 rivets in a row. Let S_t be 18,000 lb./sq. in., S_s = 12,000 lb./sq. in., S_c = 24,000 lb./sq. in. Find the strength of the joint. *Ans.* 63,600 lb.

52. In Problem 51, assume that the plates are fastened together by means of a double-strap butt joint. Each cover plate is $\frac{1}{4}$ in. thick. Find the strength of the joint. *Ans.* 67,500 lb.

53. In Problem 52, arrange the rivets in four rows of three in a row. Find the strength of the joint. *Ans.* 74,250 lb.

54. Design a lap joint to carry 70,000 lb. Use 1-in. hand-driven rivets. S_t = 16,000 lb./sq. in. (see Example III, Art. 85).

55. In a double-riveted, double-strap butt joint with a common pitch for the rivets in both rows, the continuous seam joins plates $\frac{9}{16}$ in. thick. The butt straps are $\frac{5}{16}$ in. thick. The diameter of the rivets is $\frac{3}{4}$ in. and the pitch is $2\frac{1}{2}$ in. Find the strength of the joint per linear inch of seam. What is the efficiency of the joint? Given S_s = 10,000 lb./sq. in., S_c = 20,000 lb./sq. in., and S_t = 16,000 lb./sq. in.

Ans. P_s = 17,680 lb.; P_c = 16,880 lb.; P_t = 15,750 lb. and efficiency = 63 per cent.

56. In Problem 55, let the pitch in the outer rows be 5 in., the pitch in the inner rows remaining 2.5 in. Find the efficiency.

Ans. 56.27 per cent.

57. In Problem 55, assume that the pitch is not given. Find the pitch for which the efficiency is a maximum. (Use $\frac{3}{4}$-in. rivets and $\frac{9}{16}$-in. main plates.) *Ans.* p = 2.62 in.; efficiency = 71.4 per cent.

The structural code for bridges and the A.I.S.C. code for buildings specify that the nominal diameter of the rivets shall be used when the strength of the rivets in shear and bearing is determined but that the rivet hole shall be considered to be $\frac{1}{8}$ in. larger than the nominal diameter of a rivet in computing the net width of the plate (in tension). On this basis, Problems 58 and 59 are to be solved.

58. In a lap joint the plates are 13 in. wide, and $\frac{5}{16}$ in. thick. The nominal size of the rivets is $\frac{3}{4}$ in. Ten rivets are used, three in the first row, four in the second, and three in the third. Find the strength of the joint. Given S_s = 13,500 lb./sq. in., S_c = 30,000 lb./sq. in., and S_t = 18,000 lb./sq. in. *Ans.* P_s = 59,600 lb.; P_c = 70,300 lb.; P_t = 58,400 lb.

59. The A.I.S.C. code for buildings specifies that, for power-driven rivets in double shear, S_s = 13,500 lb./sq. in. and S_c = 30,000 lb./sq. in. Take S_t = 18,000 lb./sq. in. Design a double-strap butt joint to carry a load of 112,500 lb. Use $\frac{1}{2}$-in. main plates and $\frac{3}{4}$-in. rivets (nominal size). See note preceding Problem 58.

Ans. 10 rivets. For 1-2-3-4 arrangement, b = 13.375 in.

60. A compression member is made of two 7-in. 14.75-lb. steel channels latticed together as shown (Fig. 114). The lower end of the member is held by a single pin (Fig. 114*b*). The member is subjected to a compressive force of 112,000 lb. Given S_s = 8000 lb./sq. in. and S_c = 16,000 lb./sq. in., determine the diameter of the pin and the thickness of the pin plates to obtain an economical joint. The web of a channel is 0.419 in. thick.

Ans. d = 3 in.; t = 0.456 in.

61. A boiler 4 ft. in diameter is made of steel plates $\frac{1}{2}$ in. thick. The longitudinal seams are double-riveted lap joints with rivets having a common

pitch. The diameter of the rivets is $\frac{3}{4}$ in. The pitch is $2\frac{1}{2}$ in. Given $S_s = 10{,}000$ lb./sq. in., $S_c = 22{,}500$ lb./sq. in., and $S_t = 15{,}000$ lb./sq. in., find the maximum allowable tension per linear inch of seam.

Ans. 3536 lb./in.

62. In Problem 61, what is the maximum allowable steam pressure? [See Art. 71, Eq. (9).] *Ans.* 147.3 lb./sq. in.

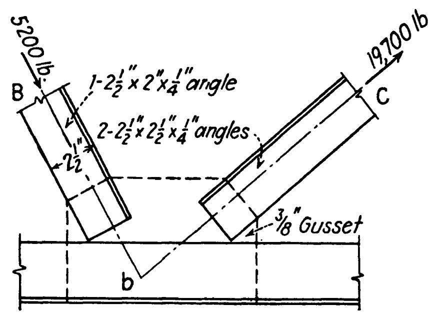


FIG. 115.

63. An engineer's flexible steel tape is 0.310 in. wide and 0.018 in. thick. In mending a break in the tape, a butt joint is made with a single strap of the same material as the tape. Four brass rivets are used on each side of the break. The rivets are 0.030 in. in diameter. The ultimate stresses are as follows: $S_s = 40{,}000$ lb./sq. in., $S_c = 60{,}000$ lb./sq. in., and $S_t = 150{,}000$ lb./sq. in. Find the pull required to break the joint. What is the ultimate efficiency?

Ans. $P_s = 113$ lb.; $P_c = 129.6$ lb.; efficiency = 13.5 per cent.

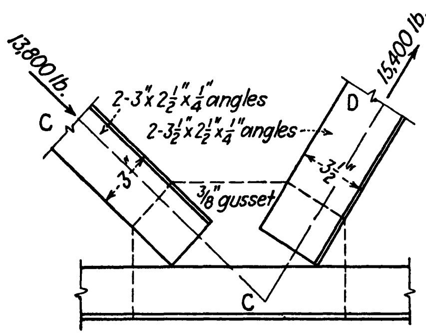


FIG. 116.

64. Figure 115 shows a joint of a steel roof truss. Member *Bb* is one $2\frac{1}{2}$- by 2- by $\frac{1}{4}$-in. angle with a compression of 5200 lb. Member *Cb* is made of two $2\frac{1}{2}$- by $2\frac{1}{2}$- by $\frac{1}{4}$-in. angles with a maximum tension of 19,700 lb. The gusset plate is $\frac{3}{8}$ in. thick. Using the unit stresses of Table I, compute the number of $\frac{5}{8}$-in. rivets required for each member. *Bb* is 2. *Cb* is 3.

65. In Fig. 116, *Cc* is made of two 3- by $2\frac{1}{2}$- by $\frac{1}{4}$-in. angles with a compression of 13,800 lb. *cD* is made of two $3\frac{1}{2}$- by $2\frac{1}{2}$- by $\frac{1}{4}$-in. angles with a tension of 15,400 lb. Using the unit stresses of Table I, compute the number of $\frac{5}{8}$-in. rivets required for each member.

66. Design the welded connections for problem 64. *Bb*, $a = 1.78$ in.; $b = 0.82$ in.; *bC*, $a = 3.51$ in., $b = 1.42$ in.

67. Design the welded connections for Problem 65.

CHAPTER V

TORSION

89. Twisting Moment. Torsion.—Figure 117 represents a cylindrical shaft or circular bar to which two equal and opposite couples PP and QQ have been applied. It is here assumed that each couple acts in a plane which is perpendicular to the axis of the shaft. Since the couples PP and QQ act in opposite directions, the part of the shaft between the two couples is twisted more or less. The shaft is therefore subjected to *torsion*

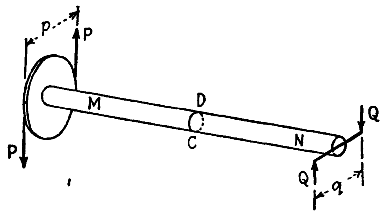


FIG. 117.

In general, if a couple acts on a shaft in a plane that is normal to the axis of the shaft, the couple tends to twist the shaft. Such a couple is a *twisting couple.* The moment of a twisting couple is called a *twisting moment* or a *twisting torque* or simply a *torque.* A torque frequently is represented by T. In Fig. 117, $T = Pp = Qq$.

For equilibrium to exist, the algebraic sum of all the torques acting on a shaft must equal zero. Thus (Fig. 117),

$$\Sigma T = Pp - Qq = 0.$$

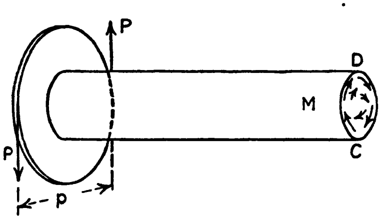


FIG. 118.

90. Resisting Moment or Resisting Torque.—Referring to Fig. 117 of the foregoing article, imagine the section CD dividing the shaft into two parts M and N. Take M free as shown in Fig. 118. Since M and N tend to rotate in opposite directions about the axis of the shaft, shearing forces are induced on the section CD as shown in the figure. If equilibrium exists, the moment (with respect to the axis of the shaft) of the shearing forces acting on the section CD must hold the *twisting moment* ($T = Pp$) in equilibrium.

The moment of the shearing forces acting on the section CD is called the *resisting moment* or the *resisting torque* or the internal torque. Hence, for equilibrium to exist,

$$\text{resisting moment} = \text{twisting moment}$$

or, as it is usually stated,

$$\text{resisting torque} = \text{twisting torque}.$$

In general, if T_r is the resisting torque acting on a section of a shaft, and T is the *resultant twisting torque* acting to the left (or to the right) of that section, then, for equilibrium to exist,

$$T_r = T.$$

91. Elements of Volume.—In the case of a circular shaft in pure torsion, for convenience of analysis the elements of volume (or simply, the elements), are assumed to be bounded by three systems of surfaces, as follows (Fig. 119):

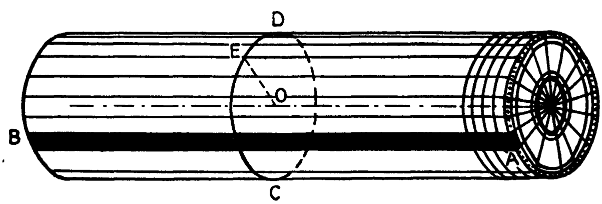


FIG. 119.

1. A system of planes at right angles to the axis, *i.e.*, a system of right sections such as CED.
2. A system of radial planes, passing through the axis of the shaft and cutting the previous sections in radial lines such as OE.
3. A system of concentric cylindrical surfaces dividing the shaft into a system of concentric thin-shelled cylinders, as shown by the circular rings on the end section. The dark-shaded ring is an instance. The outermost thin-shelled cylinder whose end is dot shaded in Fig. 119 sometimes is called the *outer skin* or the *surface layer*.

Any two consecutive surfaces of a system are supposed to be so near together that an element may be thought of as a right parallelepipedon (rectangular block).

A line of elements forming a slender prism parallel to the axis of the shaft is called a *fiber*. As here used, the word fiber has no reference to any fibrous nature of the material. A fiber in

the outer skin is an *outer fiber* and an element in the outer skin is an *outer element.* In Fig. 119, the dark-shaded fiber AB is an outer fiber, and the elements that constitute this outer fiber are outer elements.

92. Assumptions.—In the derivations of the theorems of this chapter it will be assumed

1. That cross-sections remain plane during twisting. That is, right sections such as CED (Fig. 119) *will not warp* when the shaft is subjected to pure torsion.

2. That radial lines such as OE remain straight lines during twisting.

Experiments with homogeneous *circular* shafts show that the foregoing assumptions are warranted. For noncircular shafts the foregoing assumptions are only approximately correct. If a noncircular shaft is twisted (see Art. 97), plane sections become warped. In the following discussion, if no statement to the contrary is made, it is to be understood that the shaft is circular and that the assumptions stated above are true.

93. Angle of Torsion. Helix Angle.—Let a circular shaft of length L and radius r be subjected to two equal and opposite couples, one at each end (Fig. 120). If, for convenience, it is

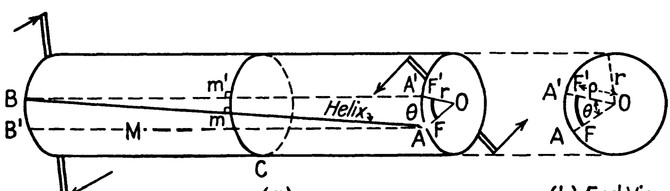


FIG. 120.

assumed that the left end is held fixed, then the twisting of the shaft consists of a rotation of the right end through an angle θ, and the point A (which was originally at A') moves through the distance

$$A'A = r\theta. \tag{a}$$

where θ is measured in radians. The angle θ is called the *angle of torsion* or the *angle of twist.*

Consider the line AB on the surface of the shaft (Fig. 120). Originally, before the shaft was twisted, AB was a straight line parallel to the axis of the shaft. That is, AB coincided with $A'B$. Now, however, owing to the twisting of the shaft, AB

takes the form of a helix. If a sheet of tracing paper is wrapped around the shaft, and the helix AB together with the lines AB' and $A'B$ are traced on the paper, it will be found when the paper is unwrapped (Fig. 121) that the helix will appear as a straight line AB making an angle δ with $A'B$ (or AB'). The angle δ is called the *helix angle.** If δ is measured in radians,

$$A'A = L\delta. \tag{b}$$

Hence, equating the two values of $A'A$, [Eqs. (*a*) and (*b*)], and solving for δ,

$$\delta = \frac{r\theta}{L}. \tag{c}$$

in which δ and θ must be measured in radians.

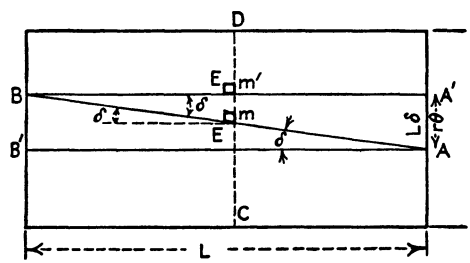


FIG. 121.

94. Shearing Stresses on the Faces of an Element.—Originally, before the couples were applied to the shaft, the element E was at m' (Fig. 121), and the angles at its corners were all right angles, but now, owing to the twisting of the shaft, the element is at m and the angles of its corners (looking toward the axis of the shaft) have changed, each by an amount δ. It is evident that this angular distortion of the element equals the helix angle of the twisted shaft. This is true only if sections of the shaft such as CD remain plane sections during the twisting. If the section becomes warped (Fig. 121*A*), the angular distortion of the element E is not necessarily equal to δ. Hence the theory of this chapter is strictly applicable only to circular (cylindrical) shafts.

In Art. 19, it was shown that the angular distortion of an element can be brought about only by shearing stresses (all of the same intensity, S_s) acting on four of its faces. In Fig. 122, the

* In a shaft under conditions of actual use, the angle δ is so small that $\tan \delta$ may be put equal to δ in radians.

element is shown on an enlarged scale and the shearing stresses on its faces are indicated by arrows, the faces being perpendicular to the plane of the paper. In Art. 26, the shearing modulus of elasticity is defined as $E_s = S_s/\delta$; so that

$$S_s = E_s\delta \tag{a}$$

provided the elastic limit is not passed.

As shown in Art. 93,

$$\delta = \frac{r\theta}{L}. \tag{b}$$

Combining Eqs. (a) and (b), we obtain

$$S_s = \frac{r\theta E_s}{L} \tag{c}$$

as the intensity of the shearing stress on the faces of an *outer* element of a shaft subjected to pure torsion, provided the elastic

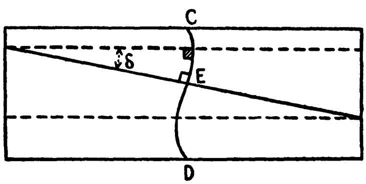


FIG. 121A.

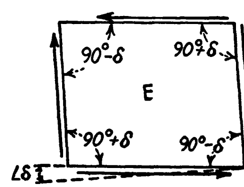


FIG. 122.

limit is not exceeded. In Eq. (c), r equals the radius of shaft, L equals the length, θ equals the angle of torsion or angle of twist, and E_s equals the shearing modulus of elasticity.

Consider now an element inside the shaft, whose distance from the axis is ρ. Let S_s' equal the intensity of the shearing stress on the faces of this element. To obtain S_s', note that the only change in the argument of Art. 93 lies in the substitution of ρ for r. Thus, for a point F, distant ρ from the axis (Fig. 120*b*), $FF' = \rho\theta$. (This is not necessarily true if radial lines, such as OFA, do not remain straight during the twisting.) Substituting ρ for r in Eq. (c), we obtain

$$S_s' = \frac{\rho\theta E_s}{L} \tag{d}$$

as the intensity of the shearing stress on the faces of an element distant ρ from the axis. Dividing Eq. (d) by Eq. (c), member by member,

$$\frac{S_s'}{S_s} = \frac{\rho}{r} \qquad \text{or} \qquad S_s' = \frac{\rho}{r}S_s. \tag{e}$$

That is, *within the elastic limit, the intensity of the shearing stress on the faces of an element of a circular shaft in pure torsion is directly proportional to the distance of this element from the axis of the shaft. Hence the maximum intensity of the shearing stress is induced on the outer elements of the shaft.*

Example.—A circular steel shaft of radius $r = 0.5$ in. and of length $L = 5$ ft. is held fixed at the left end. A torque applied to the right end rotates this end through 5.73° (Fig. 120). If $E_s = 12{,}000{,}000$ lb./sq. in. find the intensity of the shearing stress on the faces of an outer element; also on the faces of an element distant $\frac{1}{3}$ in. from the axis.

The angle of torsion (angle of twist) is

$$\theta = \frac{5.73\pi}{180} = 0.10 \text{ radian.}$$

From Eq. (c),

$$S_s = \frac{0.5 \times (0.10) \times 12{,}000{,}000}{5 \times 12} = 10{,}000 \text{ lb./sq. in.}$$

From Eq. (e),

$$S_s' = \frac{\frac{1}{3}}{\frac{1}{2}} \times 10{,}000 = 6667 \text{ lb./sq. in.}$$

on an element $\frac{1}{3}$ in. from the axis.

95. Torsion Formula. Strength in Torsion.—Referring to Fig. 120a of Art. 93, take free the part of the shaft to the left of the section CD (Fig. 123). Since equilibrium exists, the torque (moment of the twisting couple PP) must be held in equilibrium by the resisting torque (moment with respect to the axis of the shaft of all the shearing stresses) acting on the section CD. That is (Art. 90),

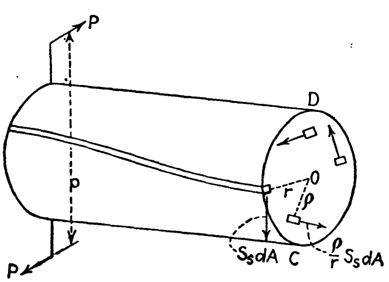


FIG. 123.

$$T = T_r.$$

To find an expression for T_r, proceed as follows: The section CD is composed of an infinite number of dA's, each dA being the area of an exposed face of an element. If S_s equals unit stress in shear on an outer element dA, distant r from O, then, $\frac{\rho S_s}{r} =$

unit shearing stress on a dA distant ρ from O [Art. 94, Eq. (e)], and $\frac{\rho S_s}{r}dA$ = shearing force acting on this dA, distant ρ from O. Therefore the sum of the moments (with respect to the axis of the shaft) of the shearing forces acting on all of the dA's that constitute the section CD is

$$T_r = \int\left(\frac{\rho S_s dA}{r}\right)\rho = \frac{S_s}{r}\int \rho^2 dA = \frac{S_s J}{r}$$

where $J = \int \rho^2 dA$,

= polar moment of inertia of the section CD with respect to the center O.

Hence, since $T = T_r$,

$$T = \frac{S_s J}{r} \tag{1}$$

provided the elastic limit is not passed.

Equation (1) is called the *torsion formula.* It is a formula for strength. In this formula, S_s is the unit shearing stress in the outer fiber and therefore is the maximum unit shearing stress induced on the section CD. Hence, for a given allowable unit shearing stress, the torsion formula gives the maximum allowable torque T that may be applied to the right (or to the left) of a section. That is, the torsion formula enables one to determine T, the torsional strength of a circular shaft.

Note.—It is necessary to be consistent in the use of units employed. If S_s is given in pounds per square inch (as is usually the case) and J and r depend upon inch units, then T will be expressed in inch-pounds of torque.

96. Polar Moment of Inertia.—The torsion formula [Eq. (1)] contains J, the polar moment of inertia of the cross-section with respect to the axis of the shaft. In the study of the moments of inertia of areas, it was seen that for a circular section,

$$J = \frac{\pi r^4}{2};$$

for a hollow circular section,

$$J = \frac{\pi}{2}(r^4 - r_1^4).$$

Hence for a solid cylindrical shaft

$$T = \frac{S_s J}{r} = \frac{S_s \pi r^3}{2} \tag{1a}$$

and for a hollow cylindrical shaft,

$$T = S_s\frac{\pi}{2}\frac{r^4 - r_1^4}{r} = \frac{S_s\pi(r^2 - r_1^2)(r^2 + r_1^2)}{2r}. \tag{1b}$$

Note that r equals the distance of *outer* element from the axis of the shaft (see Fig. 124).

Example I.—What is the safe torsional strength of a solid steel shaft of radius $r = 1.5$ in. if the safe unit shearing stress is taken as $S_s = 9000$ lb./sq. in.?

Ans. $T = S_sJ/r = 9000\pi r^3/2 = 47{,}700$ in.-lb.

Example II.—A solid steel shaft is to be subjected to a torque of 500 ft.-lb. The allowable unit shearing stress is

$$S_s = 10{,}000 \text{ lb./sq. in.}$$

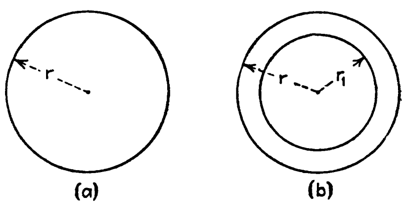


FIG. 124.

What should be the radius of the shaft? Substituting in the torsion formula [Eq. (1*a*)],

$$500 \times 12 = \frac{10{,}000\pi r^3}{2}.$$

Therefore

$$r = 0.726 \text{ in.}$$

Example III.—A hollow shaft, whose outer radius is $r = 2$ in. and whose inner radius is $r_1 = 1$ in., resists a torque of 7500 ft.-lb Find the maximum unit shearing stress induced in the shaft. Substituting in the torsion formula [Eq. (1*b*)],

$$7500 \times 12 = \frac{S_s\pi(16 - 1)}{2(2)}.$$

Therefore

$$S_s = 7640 \text{ lb./sq. in.}$$

Example IV.—For a given value of the allowable unit shearing stress S_s the torsional strength T of a solid circular shaft is proportional to the cube of its radius [Eq. (1*a*)]. Compare the torsional strengths of two circular shafts (S_s being the same for both shafts), if the radius of one is twice that of the other.

Ans. 8 to 1.

Example V.—A solid shaft has a radius of 2 in. A hollow shaft of the same length has an inner radius of 2 in.

1. Find the outer radius of the hollow shaft so that it will contain the same amount of material as the solid shaft.

$$\pi(r^2 - 2^2) = \pi 2^2.$$

Therefore

$$r = 2\sqrt{2} \text{ in.}$$

2. Compare their torsional strengths if S_s is the same for both. *Ans.* The strength of the hollow shaft is 2.12 times that of the solid shaft.

Problem 68. *a.* Design a solid circular shaft that will have the same torsional strength as a hollow shaft whose inside diameter is 10 in. and whose

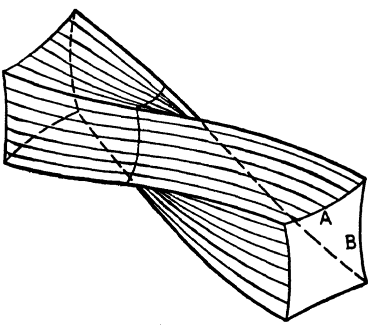


FIG. 125.

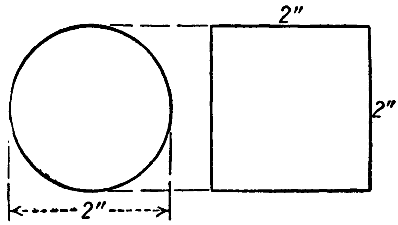


FIG. 126.

shell is $\frac{1}{2}$ in. thick. The two shafts are to be made of the same kind of material and are to be equally stressed. *Ans.* $d = 7.5$ in.

b. Compare the weights of the two shafts per linear foot.
Ans. 2.68 to 1.

97. Square Shaft.—Figure 125 shows a square shaft under torsion. Note that the cross-section is warped. It can be shown that, owing to the warping of the section, the maximum shearing stress occurs at the middle of the sides, *i.e.*, at A and B. The torsion formula is not strictly applicable, therefore, to a square shaft or to any noncircular shaft.

It has been found experimentally that the strength of a square shaft is only slightly greater (about 6 per cent) than that of a circular shaft having a diameter equal to the side of the square. For instance, a circular shaft 2 in. in diameter is very nearly as strong in torsion as a square shaft whose lateral dimensions are 2 in. (Fig. 126). Hence, in designing a square shaft, the side

of the cross-section may be made equal to the diameter of a circular shaft that can carry the same torque.

98. Torsional Stiffness.—Eliminating S_s between the two equations

$$S_s = \frac{r\theta E_s}{L}$$

(Art. 94) and

$$T = \frac{S_s J}{r}$$

(Art. 95), we obtain

$$T = \frac{JE_s\theta}{L}.$$

Or, solving for θ,

$$\theta = \frac{TL}{JE_s}.$$

Hence

$$E_s = \frac{TL}{J\theta}. \tag{2}$$

where θ is measured in radians. Since for a given torque θ the angle of torsion of a shaft of given length L may be taken as a measure of the torsional stiffness of the shaft, Eq. (2) may be called the *formula for torsional stiffness.*

Note.—There are several types of torsion-testing machines. In a good machine of this kind, the torque T and the angle of torsion θ (for a selected length of the test-piece) are recorded by suitable measuring devices. By substituting observed values of T and θ in Eq. (2), E_s, the shearing modulus of elasticity, is determined. Experiments with cylindrical test-pieces show that, for each kind of material, E_s is practically a constant within the elastic limit.

Assume now that a torque T is applied to a test-piece and that the angle of torsion θ is recorded. If the angle of torsion disappears when the torque is removed, the shearing stress is within the elastic limit. By a series of tests on a given test-piece, the maximum value of T may be found for which the torsional strain disappears when the torque is removed. That is, the value of T at the elastic limit may be found. With the value of T thus determined, the elastic limit S_s may be computed from the equation $T = S_sJ/r$.

Example I.—A steel rod is used in a torsion test. The gage-length of the rod is $L = 36$ in. and the radius of the rod is $r = 1$ in. If a torque of $T = 23{,}560$ in.-lb. produces an angle of torsion of $\theta = 2.58°$, find the shearing modulus of elasticity. Also find the unit shearing stress in the outer fiber. ($1° = 0.01745$ radian, and 1 radian $= 57.3$ degrees, approximately.)

$$E_s = \frac{TL}{J\theta} = \frac{23{,}560 \times 36}{\frac{\pi 1^4}{2} \times 2.58 \times 0.01745} = 12{,}000{,}000 \text{ lb./sq. in.}$$

$$S_s = \frac{Tr}{J} = \frac{T}{\frac{\pi}{2}r^3} = \frac{23{,}560}{\frac{\pi}{2}1^3} = 15{,}000 \text{ lb./sq. in.}$$

Example II.—In a torsion test on a steel rod of radius $r = 1.2$ in., the maximum torque within the elastic limit was found to be $T = 65{,}200$ in.-lb. Find the elastic limit in shear.

$$S_s = \frac{Tr}{J} = \frac{T}{\left(\frac{\pi r^3}{2}\right)} = \frac{65{,}200}{\frac{\pi}{2}(1.2)^3} = 24{,}000 \text{ lb./sq. in.}$$

99. Torsional Effect of a Force.—The deformation (strain) of a body subjected to external forces may be complex. A body may be elongated, bent, and twisted all at the same time.

When the strain of a body is complex, it is convenient, as a rule, to resolve the strain into a combination of simpler strains. For instance, in Fig. 127*a*, the torsional effect and the bending effect of the force P may be determined separately and these effects may then be combined to obtain the resultant effect of the force P on the shaft.*

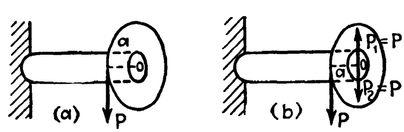


FIG. 127.

At the point O in the axis of the shaft (Fig. 127*b*), introduce two equal and opposite forces P_1 and P_2, each equal and parallel to P. This does not disturb the equilibrium of the pulley and shaft considered together as one body. The two forces P_1 and P form a twisting couple. The moment of this couple (the twisting moment, or torque) is $T = Pa$. Owing to the

* With the proviso that the elastic limit is not exceeded. If the elastic limit is exceeded, the law of proportionality (Art. 52) is not applicable.

torque T shearing stresses are induced in the shaft. The force P_2 acting down tends to bend the shaft. That is, owing to P_2 bending stresses are induced in the shaft (bending will be considered in a later chapter). As will be seen later, P_2 also induces direct vertical shearing stresses in the shaft. In most cases, however, the vertical shearing stresses due to P_2 may be neglected. The stresses in the shaft may be considered, therefore, as a combination of the stresses due to the torque $T = Pa$ and those due to $P_2 = P$.

In this chapter, only the torsional effect of forces on a shaft is under consideration. For the present it may be assumed that the bending effect of a force may be neglected. The combined effect of torsion and bending will be considered in Chap. X.

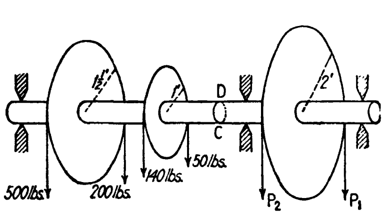


FIG. 128.

If a system of forces acts on a shaft, the algebraic sum of the moments, with respect to the axis of the shaft, of the external forces that act to one side of a section is called the *external torque* (resultant torque) for that section.

Example.—In Fig. 128, assume that the shaft rests in frictionless bearings. Assume also that each force lies in a plane that is perpendicular to the axis of the shaft. Consider the section CD. The torque of the forces to the left of the section is

$$T = (140 - 50) \times 1 + (500 - 200) \times 1.5 = 540 \text{ ft.-lb.}$$

If the shaft is in equilibrium, the torque to the left of the section must be equal and opposite to the torque to the right of the section. Hence

$$540 = (P_1 - P_2) \times 2,$$

or

$$P_1 - P_2 = 270 \text{ lb.}$$

Note.—In this chapter, all shafts will be assumed to be in equilibrium. It should be remembered, however, that equilibrium does not necessarily mean "standing still." Rest is only a particular case of equilibrium. The shaft represented in Fig. 128 will be in equilibrium even if it is rotating, provided it is rotating

with *constant speed.* If it has no change of velocity, it will not have any acceleration.

100. Power Developed by a Torque.—Figure 129 represents a pulley acted upon at a particular instant by a tangential force P as shown. Assume that at this instant the pulley is rotating at an angular velocity ω. It is required to find an expression for the power developed by the force P at the instant.

In the time dt the point A on the rim of the pulley, and therefore the point of application of the force P, moves from A to A', a distance ds. This is the distance also through which the force P "works." In this infinitesimal rotation, the work done by P is $dW = Pds$.

By definition, power is the rate of doing work. Hence, if L equals power developed by P,

$$L = \frac{dW}{dt} = P\frac{ds}{dt} = Pv \qquad (a)$$

in which v = velocity of the rim of the pulley at the particular instant.

Since at the instant the applied torque is $T = Pa$ and $v = \omega a$ Eq. (a) may be written

$$L = Pv = P\,\omega a = Pa\,\omega = T\omega. \qquad (b)$$

Expressed in words, at any given instant *power equals torque multiplied by angular velocity.*

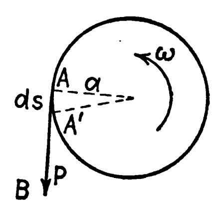


FIG. 129.

If at the particular instant the angular velocity is such that the rate of rotation is n revolutions per unit of time, $\omega = 2\pi n$. Equation (b) may then be written

$$L = 2\pi nT. \qquad (c)$$

101. Transmission of Power.—A shaft is often used as a means of transmitting power. Assume in Fig. 130 that the pulley A is driven by an engine. Pulley A is rigidly attached to the shaft and causes it to rotate. Pulley B also rotates with the shaft and is used to turn the dynamo. Hence power is transmitted from pulley A to pulley B through the shaft.

Consider pulley A. The torque applied to this pulley is $T = (P_1 - P_2)a$. If n equals the number of revolutions per unit time, the power developed is [Eq. (c), Art. 100]

$$L = 2\pi nT.$$

To reduce to horsepower, divide by N, where N equals the number of units of work that must be done in a unit of time to constitute a horsepower. (1 hp. = 550 ft.-lb. of work per second; or 6600 in.-lb./sec.; or 396,000 in.-lb./min.) The horsepower transmitted by the shaft is, therefore,

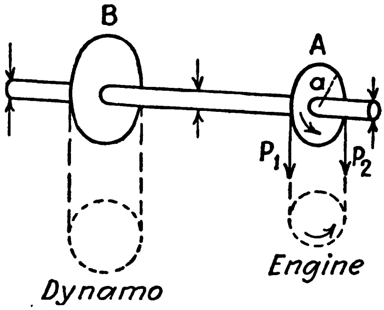


FIG. 130.

$$H = \frac{2\pi nT}{N} = \frac{Pv}{550} = \frac{T\omega}{550}.$$

Therefore

$$T = \frac{NH}{2\pi n}. \qquad (a)$$

In formula (*a*), any consistent system of units may be used. Usually the inch-pound-minute system is convenient. Then, as stated above $N = 396{,}000$.

$$\text{Therefore} \quad T = \frac{NH}{2\pi n} = \frac{396{,}000}{2\pi}\frac{H}{n} = 63{,}000\frac{H}{n} \qquad (3)$$

in which n = the number of revolutions per minute (r.p.m.),
and T = torque in inch-pounds.

102. Résumé.—In the foregoing articles of this chapter four very important formulas were developed; *viz.*,

$$T = \frac{S_s J}{r} \text{ (Art. 95)} \qquad (1)$$

$$= \frac{S_s \pi r^3}{2} \text{ (for a solid circular shaft of radius } r\text{)}$$

$$= \frac{S_s \pi (r^4 - r_1^4)}{2r} \text{ (for a hollow circular shaft, inner radius } r_1 \text{ outer radius } r\text{).}$$

$$\theta = \frac{TL}{JE_s} \text{ (Art. 98).} \qquad (2)$$

$$T = 63{,}000\frac{H}{n} \text{ (Art. 101).} \qquad (3)$$

In formula (3), H = horsepower, n = revolutions per minute (r.p.m.), and T = torque in inch-pounds.

$$H = \frac{Pv}{550} = \frac{T\omega}{550} \qquad (4)$$

In formula (4), H = horsepower, P = force acting on the rim of the pulley, in pounds, v = velocity of a point on the rim of the

pulley, in feet per second, T = torque in foot-pounds, and ω = radians per second.

Example I.—With reference to Fig. 130, $P_1 = 450$ lb.,

$$P_2 = 150 \text{ lb.},$$

$a = 1$ ft., and $n = 120$ r.p.m. What horsepower is transmitted to the shaft?

$$T = (450 - 150) \times 1 \times 12 = 3600 \text{ in.-lb.}$$

Hence by Eq. (4),

$$H = \frac{3600(4\pi)}{6600}.$$

Therefore

$$H = 6.86$$

Example II.—Solve Example I using the foot-pound-second system of units. In this system, $a = 1$, $n = 2$, and $N = 550$. Using Eq. (4),

$$H = \frac{300(4\pi)}{550}.$$

Again

$$H = 6.86.$$

Example III.—A solid cylindrical shaft is to transmit 300 hp. and is to make 150 r.p.m. The safe shearing stress is

$$S_s = 10{,}000 \text{ lb./sq. in.}$$

Required the diameter of the shaft.

$$T = \frac{300(6600)}{5\pi}$$

Substituting in the torsion formula [Eq. (1)],

$$126{,}000 = 10{,}000\frac{\pi r^3}{2}.$$

Therefore

$$r = 2 \text{ in.}; \qquad d = 4 \text{ in.}$$

Example IV.—A hollow shaft whose outer radius is $r = 3$ in. and whose inner radius is $r_1 = 2$ in. transmits 400 hp. and makes 105 r.p.m. Find the unit shearing stress induced in the outer fiber.

$$T = 400(6600)\frac{2}{7\pi} = 240{,}000 \text{ in.-lb.} = S_s\frac{\pi(r^4 - r_1^4)}{2r}.$$

Therefore

$$S = 7050 \text{ lb./sq. in.}$$

PROBLEMS

69. A steel shaft is 4 in. in diameter ($d = 4$ in.) and is rotating at a speed of 210 r.p.m. If the angle of twist is 1° in 3 ft. of length, find the horsepower the shaft is transmitting. Take $E_s = 12{,}000{,}000$ lb/sq. in. Assume that the elastic limit is not exceeded. *Ans.* 486.5.

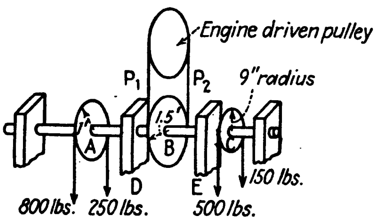


FIG. 131.

70. Find the unit stress induced in the shaft of Problem 69.

Ans. $S_s = 11{,}620$ lb./sq. in.

71. Three pulleys are fastened to a shaft (Fig. 131). Pulley B is driven by an engine. Neglect friction of bearings.

a. Assume $P_1 = 3P_2$. Find P_1 and P_2 to maintain uniform motion. *Ans.* $P_1 = 812$ lb.

b. What horsepower is delivered by the engine if the shaft turns at the rate of 180 r.p.m.? *Ans.* 27.82 hp.

c. If $S_s = 10{,}000$ lb./sq. in., required the radius of the shaft.

Ans. Radius = 0.853 in.

103. Rectangular Keys.—A pulley is fastened to a shaft usually by means of a key. Figure 132 shows a rectangular key.

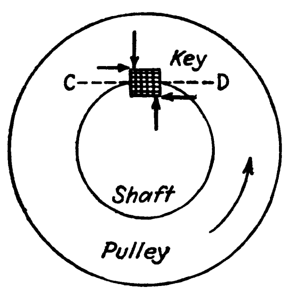


FIG. 132.

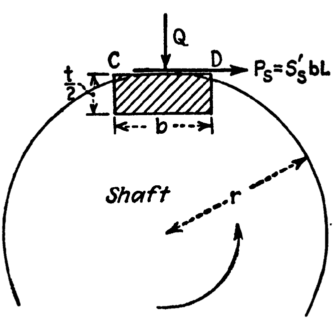


FIG. 133.

When a torque is applied to the shaft (assumed to act in a counterclockwise direction), the torque is transmitted to the pulley through the key. As a result, a shearing force P_s is induced on the section CD of the key. In Fig. 133, the shaft, together with the lower half of the key, is taken free on an enlarged scale, and the shearing force P_s on the section CD is shown.

If S_s' equals the average unit shearing stress induced on CD, L equals the length of key perpendicular to the paper, and

b equals the width of key, then $P_s = S'_s bL$. Since equilibrium exists, the moment of P_s with respect to the axis of the shaft must be equal and opposite to the torque applied to the shaft. Therefore

$$T = P_s r = S'_s bLr.^* \qquad (a)$$

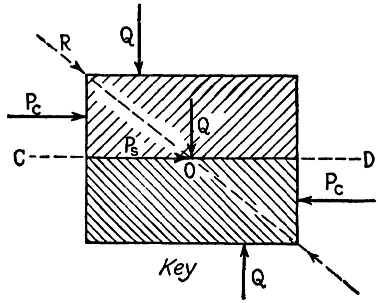


FIG. 134.

Take free the lower half of the key (Fig. 135). If the key is driven into place and fits snugly, we may assume that the bearing force P_c exerted by shaft on key is uniformly distributed over the lower half DB of the key. If S_c equals the unit bearing stress and t equals the thickness of the key,

$$P_c = \frac{S_c tL}{2}$$

in which L equals the length of the key ($\perp$ paper). Since equilibrium exists, $P_s = P_c$. Hence

$$P_s = P_c = \frac{S_c tL}{2}.$$

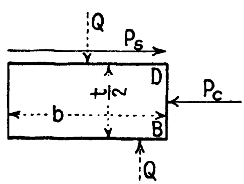


FIG. 135.

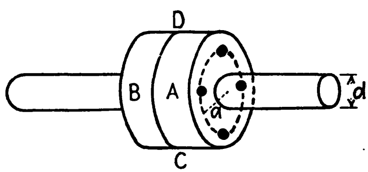


FIG. 136.

Or, since $T = P_s r$ [Eq. (a)],

$$T = \frac{S_c tLr}{2}. \qquad (b)$$

Example I.—By means of a key, a pulley is fastened to a shaft (Fig. 132). Diameter of shaft is 2 in. If 10,000 lb./sq. in. is

* There is also a normal force Q acting on the section CD (Fig. 134). This normal force, however, acts through the axis of the shaft (or very nearly so, at least) and therefore has no moment with respect to the axis of the shaft. Thus R, the resultant of P_c and Q, goes through O, the center of the key. At O, resolve R into its two components P_s and Q.

the allowable shearing stress in the shaft and 8000 lb./sq. in. is that in the key, find the minimum safe width of the key if $L = 4$ in.

The torque which the shaft can safely carry is

$$T = \frac{S_sJ}{r} = \frac{10{,}000\pi \times 1^3}{2} = 15{,}710 \text{ in.-lb.*}$$

Hence, by Eq. (*a*),

$$15{,}710 = 8000 \times b \times 4 \times 1.$$

Therefore

$$b = 0.49 \text{ in.} \qquad \text{or} \qquad \tfrac{1}{2} \text{ in.}$$

Example II.—In Example I, the allowable bearing stress between key and pulley, or key and shaft, is $S_c = 24{,}000$ lb./sq. in. Find t, the minimum safe thickness of the key.

By using Eq. (*b*),

$$15{,}710 = 24{,}000 \times t \times 4 \times \tfrac{1}{2}.$$

Therefore

$$t = 0.328 \text{ in.} \qquad \text{or} \qquad \tfrac{3}{8} \text{ in.}$$

Example III.—Keys are frequently made with a square section. Show that this should be the case if the allowable bearing stress is made equal to twice the allowable shearing stress, $S_c = 2S'_s$. By combining Eqs. (*a*) and (*b*),

$$S'_sbLr = \frac{S_ctLr}{2}.$$

Since $S_c = 2S'_s$, $b = t$.

Example IV.—The diameter of a shaft is 4 in. A 1- by $\frac{5}{8}$- by 8-in. key is used to fasten the pulley to the shaft. What unit stresses are induced in the key if the shaft is transmitting 360 hp. at 180 r.p.m.?

$$T = 63{,}000 \frac{360}{180} = 126{,}000 \text{ in.-lb. [Eq. (3)]}$$

Therefore [Eq. (*a*)]

* The keyway decreases the effective radius of the shaft. In the design of commercial keys, the effect of the keyway upon the strength of the shaft should be considered. This may be done by using a lower unit shearing stress, or a smaller diameter for the shaft. The complete design of keys, taking into account all possible refinements, is beyond the scope of this text.

$$126{,}000 = S'_s \times 1 \times 8 \times 2, \qquad \text{or} \qquad S'_s = 7880 \text{ lb./sq. in.}$$

By Eq. (*b*),

$$126{,}000 = \frac{S_c \times \frac{5}{8} \times 8 \times 2}{2} = 5S_c.$$

$$S_c = 25{,}200 \text{ lb./sq. in.}$$

Neglecting the weakening of the shaft by the keyway, what unit stress is induced in the shaft?

From $T = S_sJ/r$,

$$126{,}000 = S_s\frac{\pi 2^3}{2}.$$

Then $S_s = 10{,}030$ lb./sq. in.

104. Shaft Couplings.—Frequently two shafts are connected by means of a coupling such as is shown in Fig. 136. Each half of the coupling is keyed to the corresponding shaft, and the two halves A and B are bolted together. Evidently, for equilibrium to exist, the sum of the moments (with respect to the axis of the shaft) of the shearing forces acting on the section CD of the bolts must be equal to the torque T transmitted from A to B. If P_s is the total shearing force in one bolt and n is the number of bolts,

$$nP_sa = T \tag{a}$$

where a = radius of the bolt circle.*

If S_s is the unit shearing stress in the bolts and d is the diameter of a bolt, the shearing force in one bolt is

$$P_s = \frac{\pi d^2}{4}S_s.$$

Example.—A shaft is to transmit 180 hp. at 210 r.p.m. If the radius of the bolt circle is $a = 4$ in., and $\frac{3}{4}$-in.-diameter bolts are to be used, how many bolts are required if the allowable unit shearing stress in the bolts is $S_s = 8000$ lb./sq. in.?

From Eq. (3),

$$T = 63{,}000\frac{H}{n} = 63{,}000\frac{180}{210} = 54{,}000 \text{ in.-lb.}$$

* The radius of the bolts is small compared with the radius of the bolt circle. So it may be assumed that the resultant shear in a bolt acts at the center of its section, *i.e.*, at a distance a from the axis of the shaft.

The strength in shear of one bolt is

$$P_s = \frac{\pi(0.75)^2 8000}{4} = 3530 \text{ lb.}$$

Hence [Eq. (*a*)] $n \times 3530 \times 4 = 54{,}000$. Therefore $n = 3.8$, *i.e.*, four bolts would be needed.

105. Helical Spring or Spring Coil.—If a stiff wire is wrapped around a cylinder, the wire takes the form of a helix and the coil thus formed is called a *helical spring* or *spring coil.* The elongation (or the contraction) of such a coil is an interesting illustration of torsion, for it is due primarily to the torsion of the wire that the coil elongates or contracts. If the coil is to be used in tension, the ends usually are bent in toward the center of the coil and into a hook in such a way that the load P acts along the axis of the coil (Fig. 137). It will be assumed that the wire in a turn lies in a horizontal plane, an assumption approximately true for a closely coiled spring. (For open coiled springs, see Art. 107.) Let R equal the radius of coil (distance from center of wire to axis of coil, Fig. 137) and r equal the radius of wire.

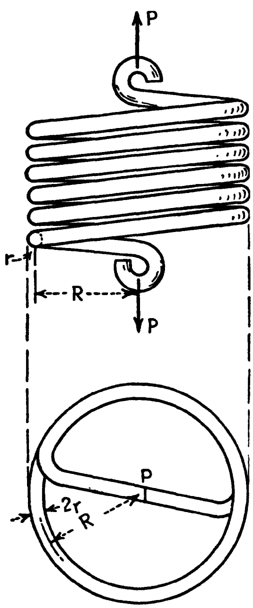


FIG. 137.

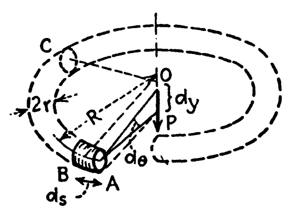


FIG. 138.

Consider an elementary length AB of the wire (Fig. 138). This element will be assumed to lie in a horizontal plane. Let ds equal the length of this element. The effect of the force P upon the element ds will be the same as if the end B were firmly held and the wire at A were bent in toward the axis of the coil, the force P being applied at O. Then the wire at A is acted upon by a torque $T = PR$. Accordingly, if S_s is the unit shearing stress induced in the wire, $S_sJ/r = T = PR$ where $J/r = \pi r^3/2$. This assumes that the wire is round. Hence

$$S_s = \frac{T}{\frac{\pi r^3}{2}} = \frac{2PR}{\pi r^3} \tag{4}$$

where R = radius of the coil.
r = radius of the wire.

106. Elongation of a closely coiled spring due to the twisting of the wire. If $d\theta$ equals angle of twist of an element ds due to the torque $T = PR$ (Fig. 138), the deflection of the point O, *i.e.*, the elongation of the coil resulting from the twisting of the given element through an angle $d\theta$ is

$$dy = Rd\theta. \tag{a}$$

In the formula $\theta = TL/JE_s$ [Eq. (2)], put $d\theta$ for θ, ds for L, PR for T, and $\pi r^4/2$ for J, and obtain

$$d\theta = \frac{2PRds}{\pi r^4 E_s}.$$

Substituting in Eq. (a),

$$dy = Rd\theta = \frac{2PR^2ds}{\pi r^4 E_s}.$$

For a complete turn of the wire,

$$y = \frac{2PR^2}{\pi r^4 E_s}\int_0^{2\pi R} ds = \frac{4PR^3}{r^4 E_s}.$$

If the coil contains n turns of wire, the total elongation of the coil is

$$e = \frac{4nPR^3}{r^4 E_s}. \tag{5}$$

Since e is directly proportional to P, the law of proportionality holds for a closely coiled spring. Hence if e_1 is the elongation due to P_1 acting alone, and e_2 is the elongation due to P_2 acting alone, then the elongation due to $P = P_1 + P_2$ is $e = e_1 + e_2$, provided the elastic limit is not exceeded.

Example.—A closely coiled spring consists of 25 turns ($n = 25$) and is made of a steel wire 0.25 in. in diameter ($r = 0.125 = \frac{1}{8}$ in.). The radius of the coil is $R = 1$ in. A load of 30 lb. ($P = 30$) is applied. Find the elongation of the coil and the unit shearing stress induced in the wire. Take $E_s = 12{,}000{,}000$ lb./sq. in.

$$e = \frac{4 \times 25 \times 30 \times 1^3}{\left(\frac{1}{8}\right)^4 \times 12{,}000{,}000} = 1.024 \text{ in.}$$

$$S_s = \frac{T}{\frac{\pi r^3}{2}} = \frac{2 \times 30 \times 1}{\pi\left(\frac{1}{8}\right)^3} = 9790 \text{ lb./in.}^2$$

107. Open Coiled Spring.*—In the foregoing article it was assumed that the wire in a turn of the coil lies in a horizontal plane. This is approximately true for closely coiled springs. Consider now an open coiled spring (Fig. 139). If R is the radius of coil and p is the pitch of coil in its strained or final state

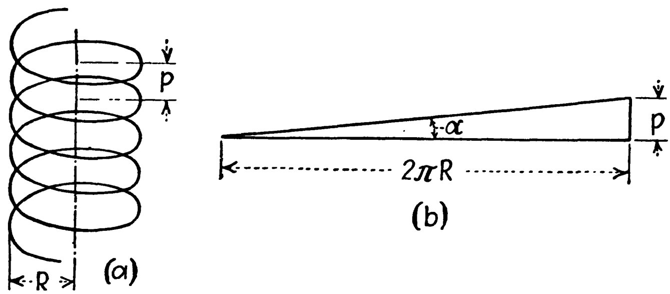


FIG. 139.

(distance between turns, center to center of wire), then the angle which the wire makes with a horizontal plane is (Fig. 139b),

$$\alpha = \tan^{-1}\frac{p}{2\pi R}. \qquad (a)$$

If the angle α is considered, it can be shown that the elongation of the coil is

$$e = \frac{4nPR^3}{r^4E_s}\left(1 + 2\frac{E_s}{E}\tan^2\alpha\right)\cos\alpha \qquad (b)$$

where E = modulus of elasticity in tension (or compression).
E_s = modulus of elasticity in shear.

Note that if $\alpha = 0$ Eq. (b) becomes

$$e = \frac{4nPR^3}{r^4E_s} \qquad (c)$$

the same as that for a closely coiled spring (Art. 106).

If $\alpha = 15°$, the elongation as given by Eq. (b) is about 2 per cent greater than that given by Eq. (c); and if $\alpha = 30°$, the elongation is about 10 per cent greater. Ordinarily, α is less than 15°.

* See MAURER and WITHEY, "Strength of Materials."

108. Note 1.—The elongation of a spring coil, as given in Art. 106 (or Art. 107), does not include the deflection of the hook portion of the coil (see Fig. 137). As a rule, the deflection of the hook portion may be neglected.

Note 2.—In Fig. 140, OB represents a horizontal rod firmly held at B. The arm OD is perpendicular to the rod. For convenience, assume OD in a horizontal position. At D a vertical load P is applied. In Art. 99 it is shown that, owing to the force P, the rod will *twist* and also *bend*. (Assume that the bending is slight so that the rod does not deviate appreciably from the horizontal position.) The amount of twist is proportional to the torque PR. It will be shown later that the curvature of the rod at a section C will be proportional to the bending moment Px, where x equals the perpendicular distance of the line of action of P from the plane of the section at C.

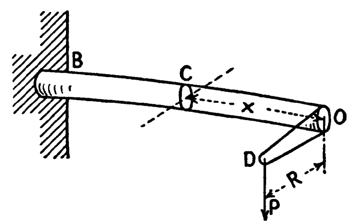


FIG. 140.

Referring now to Fig. 138, note that the line of action of P lies in the plane of every right section of the wire so that x equals zero for every section. (This is strictly true only if the turn lies in a horizontal plane.) Hence the bending moment at every section of the wire is zero. The wire, therefore, will not bend, and no deflection of the spring is caused by bending action of the load.

In an open coiled spring, P cannot be assumed to lie in the plane of a right section of the wire. Assume Fig. 138 to represent a turn of an open coiled spring making an angle α with the horizontal plane. For a given section C, P may be resolved into two components. Since a right section perpendicular to the wire is not a vertical section but makes an angle of α with the vertical, one component of P lies in the plane of the section and has the value $P_1 = P \cos \alpha$. The other component is perpendicular to the inclined section and is of amount $P_2 = P \sin \alpha$. The first component, P_1, lying in the plane of the inclined section will not bend the wire. The second component, P_2, has a moment P_2R and will exert a bending action on the wire of the coil in the plane of the turn. This bending moment will *unwind* the coil to a slight extent. If a spring coil is stretched so that it

becomes an open coiled spring (assume $\alpha = 30°$), it will be noticed that the coil unwinds slightly.

In a somewhat closely coiled spring, α is small. Then P_2, *i.e.*, $P \sin \alpha$, becomes very small and its effect may be neglected. In general, *in an ordinary coiled spring, the elongation of the coil due to the bending of the wire is negligible.*

Note 3.—Take free the part of the coil below the section at C (Fig. 141). Since equilibrium exists, a vertical force $V = P$ must act on the section at C. Owing to this vertical force, shearing stresses are induced in the elemental prisms of the wire. A shearing stress always distorts the angles of an elemental prism. Hence the coil will elongate owing to the shear induced in the wire. In the case of a closely coiled spring it can be shown that

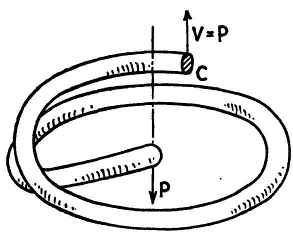


FIG. 141.

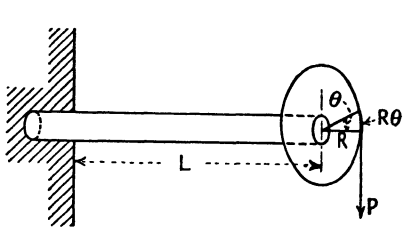


FIG. 142.

if e' equals the elongation of the coil due to the vertical shear in the wire, and if e equals the elongation due to the torsion of the wire,

$$\frac{e'}{e} = \frac{5r^2}{9R^2} \text{ (very nearly).}$$

The elongation due to vertical shear, as a rule, may be neglected.

Note 4.—The spring coils considered were assumed to hang vertically. This was done merely for convenience. Spring coils may take any position.

Example.—If $r = \frac{1}{16}$ in. and $R = \frac{1}{2}$ in., $e'/e = 0.0087$. That is, the elongation due to vertical shear in this case is less than 1 per cent of that due to torsion.

109. Torsional Resilience.—Let the force P (Fig. 142) be gradually applied (increasing gradually from zero to its final value P), and let θ equal angle of torsion of the shaft. Within the elastic limit, the work done in twisting the shaft is

$$U = \text{average force} \times R\theta = \frac{PR\theta}{2} = \frac{T\theta}{2}. \qquad (a)$$

That is, U equals the energy spent on the shaft by a torque that twists the shaft through an angle θ. Since the elastic limit is not exceeded, this energy is stored up in the shaft in the form of potential (elastic) energy and is returned when the torque is removed. The potential energy thus stored up in the shaft is called *torsional resilience* (see Art. 65).

The torsional resilience of a circular shaft under torsion may be found as follows. Within the elastic limit,

$$\theta = \frac{TL}{JE_s} \text{ [Eq. (2) Art. 102].}$$

Substituting in Eq. (a) above,

$$U = \frac{T\theta}{2} = \frac{T^2L}{2JE_s} = \text{torsional resilience.}$$

Or, since $T = S_sJ/r$,

$$U = \frac{JS_s^2L}{2r^2E_s}.$$

For a solid circular section,

$$J = \frac{\pi r^4}{2}.$$

Therefore

$$U = \frac{S_s^2 \times \pi r^2 \times L}{4E_s} = \frac{S_s^2}{4E_s} \times \text{volume of shaft.*} \qquad (b)$$

The torsional resilience (energy that can be stored up) *per unit volume* of the shaft, when the shaft is stressed up to its elastic limit, is called the *modulus of torsional resilience.* From Eq. (b) it is evident that for a circular shaft, if S_s' equals the stress at elastic limit in torsion,

$$\frac{S_s'^2}{4E_s} = \text{modulus of torsional resilience.}$$

It is sometimes advantageous to attack a problem in torsion from the standpoint of the *energy load, i.e.*, from the standpoint of the amount of energy a body can or must absorb in torsional elastic strain.

* For a hollow circular shaft (with r = outer radius, r_1 = inner radius), the resilience is

$$U = \frac{S_s^2}{4E_s} \times \text{volume of shaft} \times \frac{r^2 + r_1^2}{r^2}.$$

Example I.—A shaft of length $L = 10$ ft. has a radius of $r = \frac{1}{2}$ in. What is the maximum amount of energy this shaft can absorb in torsion if the unit shearing stress is not to exceed $S_s = 10{,}000$ lb./sq. in.?

Take $E_s = 12{,}000{,}000$ lb./sq. in.

$$U = \frac{10{,}000^2}{4 \times 12{,}000{,}000} \times \pi\left(\frac{1}{2}\right)^2 \times 120 = 196.4 \text{ in.-lb.}$$

Example II.—A bumper spring is made of 10 turns of a steel rod that is 2 in. in diameter ($r = 1$ in.). The radius of the coil is $R = 6$ in. A car when moving at the rate of 4.4 ft. per second (3 m.p.h.) bumps against this spring. What is the maximum weight the car may have if the elastic limit of the rod, namely $S_s = 24{,}000$ lb./sq. in., is not to be exceeded?

If G is the weight of the car, the kinetic energy of the car is

$$E_K = \frac{Gv^2}{2g} = \frac{G(4.4)^2}{2(32.2)} = 0.30G \text{ ft.-lb. or } 3.60G \text{ in.-lb.}$$

Therefore [Eq. (*b*)], since $L = 2\pi Rn = 2\pi \times 6 \times 10 = 120\pi$ in.,

$$3.60G = \frac{24{,}000^2}{4 \times 12{,}000{,}000} \times \pi 1^2 \times 120\pi.$$

Therefore

$$G = 3950 \text{ lb.}$$

Example III.—In Example II find the contraction of the spring.

$T = PR = S'_s \times \frac{1}{2}\pi r^3.$ Also $S'_s = 24{,}000$ lb./sq. in.

Therefore

$$P = \frac{24{,}000 \times 0.5\pi\, 1^3}{6} = 2000\pi \text{ lb.}$$

Assuming the spring closely coiled,

$$e = \frac{4 \times 10 \times 2000\pi \times 6^3}{1^4 \times 12{,}000{,}000} = 4.52 \text{ in.}$$

Hence, since there are 10 turns, the contraction per turn is 0.452 in. Since the rod is 2 in. in diameter and contact between turns must be avoided, the minimum pitch of the coil in its unstrained state is 2 in. + 0.452 in. or, say, 2.5 in.

PROBLEMS

72. What shearing stress is induced in a shaft 6 in. in diameter if the torque on the shaft is 40,000 ft.-lb.? *Ans.* 11,320 lb./sq. in.

73. A shaft must carry a torque of 2100 ft.-lb. The allowable unit shearing stress is 12,000 lb./sq. in. Find the radius of the shaft.

Ans. $r = 1.10$ in.

74. If the shaft of Problem 72 is 15 ft. long, what is the angle of twist? Take $E_s = 12{,}000{,}000$ lb./sq. in. *Ans.* $\theta = 3.24°$.

75. A steel rod, $\frac{1}{4}$ in. in diameter, is to be twisted through an angle of 90°. If the shearing stress is not to exceed 12,000 lb./sq. in., what is the minimum length the rod may have? *Ans.* 16.3 ft.

76. A shaft, 6 in. in diameter, is to transmit 1000 hp. with an allowable stress of 10,000 lb./sq. in. Compute the speed. Do not use Eq. (3), Art. 102, but employ the fundamental conception of power developed by a torque. Read carefully Art. 101. *Ans.* $n = 2.48$ r.p.s. or 149 r.p.m.

77. Use Eq. (3) and check the results of Problem 76.

78. A shaft 3 in. in diameter is rotating at a speed of 180 r.p.m. Compute the horsepower this shaft may transmit if S_s is 9000 lb./sq. in. Do not use Eq. (3), Art. 102. *Ans.* 136.

79. Use Eq. (3) and check the result of Problem 78.

80. Design a hollow cylindrical shaft that has the same torsional strength as a solid shaft 5 in. in diameter. Compare the sectional areas of the two shafts. Assume an outer diameter for the hollow shaft.

81. What is the greatest horsepower a steel shaft 3 in. in diameter can transmit at 270 r.p.m. if the allowable shearing stress is 12,000 lb./sq. in.?

Ans. 272.

82. What horsepower can the shaft of Problem 81 transmit if the allowable twist is 1° in 27 in.? *Ans.* 265.

83. A pulley B is keyed to one end of a horizontal steel shaft at the other end of which there is a drum C. Distance between pulley and drum, center to center, is 15 ft. The diameter of the pulley is 1 ft. and the diameter of the drum is 2 ft. The shaft is 3 in. in diameter. A wire whose sectional area is 0.15 sq. in. is wound round the drum and then extends 250 ft. vertically downward. The lower end of the wire is attached to a body weighing 3000 lb. Initially, the body rests on a floor and the wire is taut but not stretched. (*a*) What force P must be exerted tangentially on pulley B to lift the body from the floor? (*b*) Through what angle θ must pulley B rotate to cause the body weighing 3000 lb. to be just free from the floor? Consider both the effect of the twisting of the shaft and also the elongation of the wire.

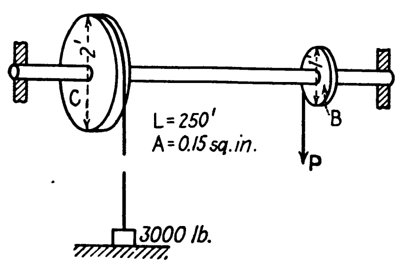


FIG. 143

$$E_s = 12{,}000{,}000 \text{ lb./sq. in.}, \; E = 30{,}000{,}000 \text{ lb./sq. in.}$$

Ans. $P = 6000$ lb.; $\theta = 0.269$ radian $= 15.4°$.

84. In Problem 83, what is the stress in the shaft? In the wire?

Ans. 6800 lb./sq. in.; 20,000 lb./sq. in.

85. How much energy is absorbed by the shaft? How much by the wire?
Ans. 1225 in.-lb.; 3000 in.-lb.

86. A pulley 3 ft. in diameter is keyed to a shaft 3 in. in diameter. Two rectangular keys of equal size are used. A force of 2000 lb. is exerted tangentially at the rim of the pulley. The pulley rotates with a speed of 120 r.p.m. (*a*) Find the horsepower transmitted by the pulley to the shaft.
Ans. 68.5 hp.

(*b*) Find the unit stress induced in the shaft. *Ans.* 6780 lb./sq. in.

(*c*) Find the total shearing stress induced in each of the two keys (see Fig. 133). *Ans.* 12,000 lb.

(*d*) If the allowable unit shearing stress in a key is 12,000 lb./sq. in. and the allowable bearing stress on the key is 24,000 lb./sq. in., what are the minimum dimensions each key should have? Assume each key to be 4 in. long. *Ans.* 0.25 by 0.25 in.

87. A closely coiled spring consists of 20 coils or turns and is made of heat-treated spring wire 0.30 in. in diameter. The radius of the coil is $R = 1.50$ in. What is the maximum load that may be hung from the end of the spring if $S_s = 50{,}000$ lb./sq. in.? *Ans.* 176 lb.

88. What will be the total elongation of the spring of Problem 87 due to a load of 150 lb. hung from its lower end? Take $E_s = 12{,}000{,}000$ lb./sq. in.
Ans. $6\frac{2}{3}$ in.

89. It requires a force of 175 lb. completely to close the spring of Problem 87. That is, a compressive force of 175 lb. will cause the coils of the spring just to touch. Find the pitch of the spring. What is the total contraction of the spring?

Note.—The pitch of a spring is the distance between two neighboring coils, center to center of wire. The free space between the two coils is the pitch minus the diameter of the wire. Assume that there are as many free spaces as there are coils. *Ans.* 0.69 in.; 7.78 in.

90. The constant C of a coiled spring is the force required to elongate or contract the coil 1 in. If the spring elongates or contracts s in., the force exerted on the spring is Cs lb. Find the constant of the spring of Problem 87 if it requires 150 lb. to contract the spring $6\frac{2}{3}$ in. What force is required to contract the spring 5 in.? *Ans.* 22.5; 112.5 lb.

91. The spring of Problem 87 is placed in a vertical position with the lower end resting on a firm support. The pitch of the spring is $p = 0.69$ in. A 15-lb. body is dropped upon the spring from a height h in. above the original position of the upper end of the spring. The constant of the spring is $C = 22.5$ lb. per inch of contraction. Find h so that the maximum contraction of the spring is 4 in. (see Art. 63, Fig. 69). *Ans.* 8 in.

92. Use the equation of dynamics, $\Sigma F = Ma$, and solve Problem 91.

Suggestion.—Find the velocity of the body as it hits the spring. Next assume the spring to have contracted a distance s. Take free the body representing the forces acting upon it, *i.e.*, the weight of the body and the variable upward force exerted by the spring on the body. Put $\Sigma F = Ma$ and find a as a function of s. Finally, use $v\,dv = a\,ds$ and find s when the velocity of the body is zero.

CHAPTER VI

SIMPLE BEAMS

SHEAR AND MOMENT

110. Beam Defined.—A member is called a *beam* if its principal function is to sustain forces that act transversely to its long dimension. Only simple beams will be considered in this chapter. As an illustration of a simple beam, consider the rectangular prismatic beam shown in Fig. 144. There are five external forces acting on the beam, the three loads P_1, P_2, and P_3 and the two reactions R_1 and R_2. The weight of the beam will be neglected for the present. The five forces are represented as concentrated forces. That is, each force represents the resultant of a force

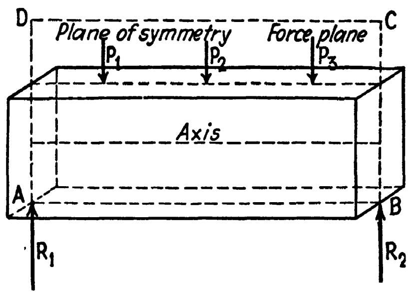


FIG. 144.

distributed over a small length of the beam. All forces are represented as acting perpendicular to the axis of the beam and as lying in a plane of symmetry (the plane $ABCD$). Note that this plane contains the axis of the beam.

The five forces P_1, P_2, P_3, R_1, and R_2 form a *system of parallel forces in a plane.* For such a system there are two independent equations of equilibrium; *viz.,* Σ moments $= 0$ and $\Sigma F_y = 0$; or two moment equations may be used. The two equations suffice to determine the two reactions R_1 and R_2 in terms of P_1, P_2, and P_3.

Illustration. Fig. 145:

$$\Sigma M_A = 0;$$
$$R_2 \times 12 - 6 \times 4 = 0.$$

Or

$$R_2 = 2 \text{ tons.}$$

$$\Sigma M_B = 0$$

$$R_1 \times 12 - 6 \times 8 = 0.$$

Or

$$R_1 = 4 \text{ tons.}$$

$$\Sigma Y = 0, \qquad 4 + 2 - 6 = 0 \qquad \text{(Check.)}$$

111. Statically Determinate and Statically Indeterminate Beams.—When the principles of statics suffice to determine the reactions, a beam is said to be *statically determinate.* Thus the beam of Fig. 145 is statically determinate since there are two reactions and two independent equations of equilibrium to determine them. When the principles of statics do not suffice to determine the reactions, the beam is said to be *statically indeterminate.* For instance (Fig. 146), if a beam rests on three supports (loads and reactions vertical), there will be three reactions, R_1, R_2, and R_3. For a system of parallel forces in a plane there are, however, *only two independent equations of equilibrium.* Hence in the case of a beam resting on three supports there are three unknown reactions and only two independent equations of statics to determine them. Such a beam is statically indeterminate. Their analysis will be presented in later chapters.

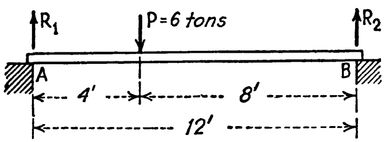


FIG. 145.

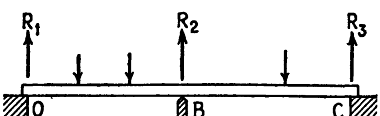


FIG. 146.

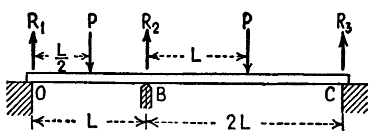


FIG. 147.

Note.—At first sight, it may seem that three equations can be found (by statics) to determine R_1, R_2, and R_3 (Fig. 146). Thus $\Sigma F_y = 0$, $\Sigma M_O = 0$, $\Sigma M_B = 0$; or $\Sigma M_O = 0$, $\Sigma M_B = 0$, $\Sigma M_C = 0$. It can be shown, however, that only two of the three equations in either group are independent. If $\Sigma F_y = 0$ and $\Sigma M_O = 0$ are chosen as the two equations, all other equations will be satisfied identically and therefore will not help to determine the reactions. This will be illustrated in the following case (Fig. 147):

$\Sigma F_y = 0$ gives

$$R_1 + R_2 + R_3 - 2P = 0. \tag{a}$$

$\Sigma M_O = 0$ gives

$$R_2L + R_33L - \frac{PL}{2} - P2L = 0;$$

or

$$R_2 + 3R_3 - \frac{5P}{2} = 0. \tag{b}$$

Subtract Eq. (*b*) from Eq. (*a*); *i.e.*, eliminate R_2;

$$R_1 - 2R_3 + \frac{P}{2} = 0. \tag{c}$$

Now take $\Sigma M_B = 0$.

$$R_1L - \frac{PL}{2} - R_32L + PL = 0.$$

Or

$$R_1 - 2R_3 + \frac{P}{2} = 0 \tag{d}$$

Equation (*d*) is the same as Eq. (*c*). Hence $\Sigma M_B = 0$ may be obtained merely by combining $\Sigma F_y = 0$ and $\Sigma M_O = 0$. Accordingly $\Sigma M_B = 0$ is not an independent equation. In general, if there are n unknowns, there must be n independent equations to determine them. For an equation to be independent, it must not be possible to obtain this equation by combining two or more of the other equations algebraically.

112. Simple Beam Defined.—Figure 144 represents one type of simple beam. Any beam that satisfies the following description will be called a *simple beam.*

1. A prism that is homogeneous and originally straight, or nearly so.
2. Relatively long when compared with its lateral dimensions.
3. The external forces (loads and reaction) acting on the beam all lie in a plane of symmetry. That is, *the load plane* (force plane) *is a plane of symmetry* (Fig. 144).

Note.—A beam is said to be *symmetrically loaded* when the load plane is a plane of symmetry.*

4. The external forces all act at right angles to the axis of the beam.

* See Art. 154 for unsymmetrical loading.

5. The bending of the beams is slight and takes place in the direction of the forces producing it.

6. The beam is statically determinate. That is, the reactions are so few in number that they can all be determined by the principles of statics alone.

In spite of the restrictions thus placed upon the beam, results will be obtained that are applicable to more complicated beams—beams for which one or more of the foregoing assumptions are modified or removed.

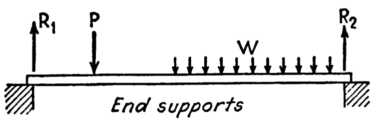


FIG. 148.

Note.—As a rule, beams will be represented as *horizontal with the loads acting vertically.* It should be remembered, however, that beams may take any position. In practice, beams frequently are vertical and the loads frequently are horizontal.

113. Types of Simple Beams.—There are three types of simple beams.

1. The beam on two *end* supports (Fig. 148).
2. The beam on two supports with one or both ends *overhanging* (Fig. 149).
3. The *cantilever* (Fig. 150).

Note that in each case there are two reactions, R_1 and R_2.*

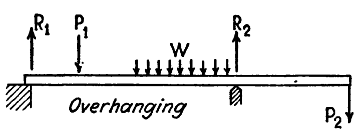


FIG. 149.

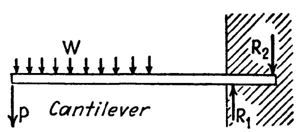


FIG. 150.

114. Distributed Load.—If a beam carries a load W uniformly distributed over a length L of the beam, the *rate of loading* is

$$w = \frac{W}{L}.$$

That is, w equals the load carried per unit of length of the beam. Thus, if a load of 24,000 lb. is uniformly distributed over a length of 12 ft., then $w = 2000$ lb./ft. of beam.

If the load is nonuniformly distributed, the rate of loading is variable. If dW equals the load on an infinitesimal length dx

* Some authors apply the term "simple beam" only to the first type, *i.e.*, to the beam on two end supports as in Fig. 148.

at the point C of the beam (Fig. 151), then the rate of loading at C is

$$w = \frac{dW}{dx}.$$

The rate of loading at C, *viz.*, w, is simply what would be the load on a unit length of the beam (shown by the dotted area in the figure) if over this unit length the load were distributed in the same way as it is over an infinitesimal length dx at C.

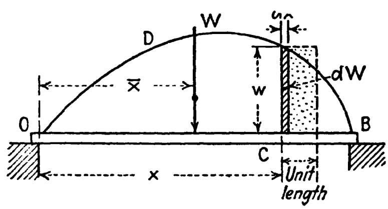


FIG. 151.

Given a nonuniformly distributed load W. This load is conveniently shown by representing the rate of loading of all points along the beam (Fig. 151). If w is the rate of loading at C, the load on a length dx at C is

$$dW = wdx = dA$$

where dA = area of the dark-shaded rectangle.

Therefore

$$W = \int_O^B wdx = A$$

where A is the area of the figure ODB. Hence, by calling the figure ODB the *rate figure*,* *the distributed load W is numerically equal to the area of its rate figure.*

Let x = distance, from point O, of the point of application of the resultant of the distributed load W (Fig. 151). In Elementary Mechanics it is shown that

$$\bar{x} = \frac{\int xdW}{W}. \qquad (a)$$

Since $W = A$ and $dW = wdx = dA$, Eq. (a) may be written

$$\bar{x} = \frac{\int xdA}{A}. \qquad (b)$$

Equation (b) gives the x-coordinate of the centroid (center of gravity) of the rate figure ODB. Hence *the resultant of a distributed load acts through the centroid of the rate figure.*

* The "rate figure" must not be confused with the "stress figure" (see Art. 9). If q equals intensity of the load at C (Fig. 151), and b equals width of surface, then $w = bq$.

115. Reactions.—Frequently the first step in the solution of a beam problem is the determination of the reactions. The student is supposed to know how to find the reactions of a simple beam. To refresh his memory, two illustrations will be given.

Example I. Fig. 152.—The beam carries a uniform load of 14 tons and a concentrated load of 2 tons, as shown in the figure. The weight of the beam is neglected. Required to find the reactions R_O and R_B.

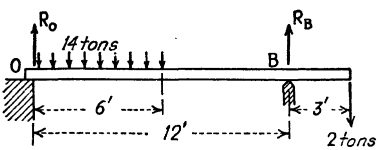


FIG. 152.

When finding the *moment* of a distributed load, this load may be imagined to be replaced by its resultant (Art. 114). Hence, by taking the whole beam free, and counterclockwise plus, $\Sigma M_O = 0$ gives

$$R_B 12 - 2 \times 15 - 14 \times 3 = 0.$$

Therefore

$$R_B = 6 \text{ tons}.$$

$\Sigma M_B = 0$ gives (taking clockwise as plus)

$$R_O 12 - 14 \times 9 + 2 \times 3 = 0;$$

or

$$R_O = 10 \text{ tons}.$$

As a check use $\Sigma F_y = 0$ and see if R_O and R_B satisfy this condition, as they should if no errors were made. $10 + 6 - 14 - 2 = 0$.

Example II. Fig. 153.—A simple beam on two end supports carries a uniformly distributed load $W_1 = 4000$ lb. and a triangularly distributed load $W_2 = 6000$ lb. Required to find the reactions.

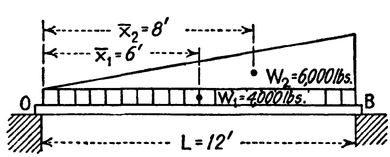


FIG. 153.

$\Sigma M_O = 0$ gives

$$R_B \times 12 - 4000 \times 6 - 6000 \times 8 = 0;$$

Therefore

$$R_B = 6000 \text{ lb}.$$

$\Sigma M_B = 0$ gives

$$R_O \times 12 - 4000 \times 6 - 6000 \times 4 = 0;$$

or

$$R_O = 4000 \text{ lb.}$$

As a check see if $\Sigma F_y = 0$. $6000 + 4000 - 4000 - 6000 = 0$.

116. Internal or Resisting Moment.—A loaded beam tends to bend more or less. If the beam is slender, the bending is quite noticeable. If the beam is stiff and short, the bending is not perceptible to the naked eye but may be detected by means of delicate measuring devices.

In the beam of Fig. 154 (shown as rectangular for convenience), CE and GI are two sections a distance pn apart. Assume now

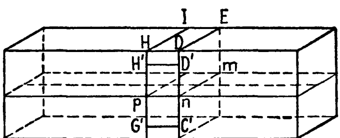


FIG. 154.

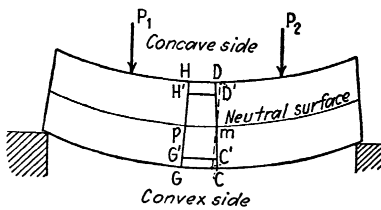


FIG. 155.

that the beam is loaded (beam originally horizontal, loads and reactions vertical, Fig. 155). Experiments warrant the assumption that within the beam there is a surface pm, called the *neutral surface*, such that, when the beam is bent (Fig. 155), all fibers (as $D'H'$) above the neutral surface are contracted and all fibers (as $G'C'$) below the neutral surface are elongated; while fibers (such as pn) along the neutral surface are neither contracted nor elongated. That is, every fiber on the concave side of the neutral surface is subjected to an axial compressive stress and every fiber on the convex side is subjected to an axial tensile stress.

Imagine the bent beam cut by the section CDE and take the part to the left of this section free, thus exposing the cross-section CDE (Fig. 156). Represent the normal force on an elementary area dA by dF. There is also a tangential force (shear) acting on this elementary area (as will be seen later), but for the present we are concerned only with the normal force. Note that the elementary area dA may be thought of as the end of a fiber (Fig. 156). Evidently, the force acting on an elementary area is

simply the action of the part to the right of the section CDE on the part to the left of this section.

Consider now the normal forces dF acting on the elementary areas dA. Since all the fibers above the neutral surface pm are in compression, the normal forces on all the dA's above the line mn are compressive forces. In like manner, the normal forces on all the dA's below the line mn are tensile forces. Since fibers along the neutral surface are neither in compression nor in tension, there are no normal forces acting on elementary areas along the line mn. Hence the line mn may be called the *neutral axis* of the section.

The normal forces acting on the dA's above the neutral axis, being all compressive, may be combined into a resultant

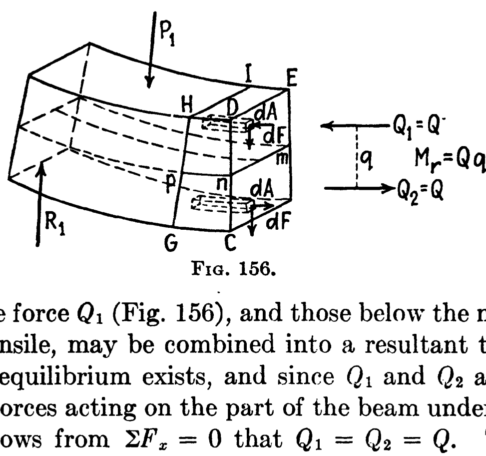


FIG. 156.

compressive force Q_1 (Fig. 156), and those below the neutral axis, being all tensile, may be combined into a resultant tensile force Q_2. Since equilibrium exists, and since Q_1 and Q_2 are the only horizontal forces acting on the part of the beam under consideration, it follows from $\Sigma F_x = 0$ that $Q_1 = Q_2 = Q$. That is, Q_1 and Q_2 constitute a *stress-couple* whose moment is

$$M_r = Qq.^*$$

In general, the system of normal forces acting on a section of a beam subjected *only to transverse loads* is equivalent to a stress-couple. The moment of this stress-couple will be designated by M_r. For reasons that will appear later, the internal moment M_r acting on a section of a beam will be called the *resisting moment*.

117. Internal or Resisting Shear.—Imagine the beam divided into two parts A and B by the section CD (Fig. 157). Owing

* Since the beam is assumed to satisfy the conditions of Art. 112, the stress-couple Q_1Q_2 lies in the plane of symmetry containing the loads and the reactions.

to the external forces acting on the beam, the two parts A and B tend to slide one on the other along their surface of contact CD. Hence shearing forces are induced on the elementary areas dA as indicated in Fig. 156. Since these shearing forces are all parallel and all lie in a plane (the plane of the section), they may be combined into a resultant. The resultant shear acting on a section of a beam will be designated by V_r and will be called the internal or *resisting shear.**

Note.—It is assumed that the beam will not twist. If the beam is symmetrically loaded (force plane being a plane of symmetry of the section), the beam will not twist.

118. Free Body Sketches.—Referring to Fig. 157, take A free (Fig. 158*a*). Note that the internal forces acting on the section CD are represented by a stress-couple (the resisting couple

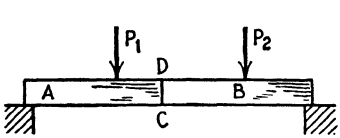


FIG. 157.

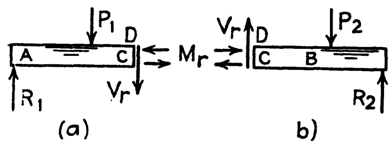


FIG. 158.

whose moment is M_r) and a vertical force (the resisting shear V_r). Hence, action and reaction being equal and opposite, if B is taken free (Fig. 158*b*), the M_r and V_r acting on B must be represented as equal and opposite to the M_r and V_r acting on A.

Note.—It will be seen later that the resisting shear V_r and the resisting moment M_r acting on a section play very important roles in the study of beams. It is necessary therefore to be able to determine the values of V_r and M_r at any section. The next article will show how this is done.

119. Vertical Shear. Bending Moment.—Consider the beam shown in Fig. 159*a*. Imagine the section CD dividing the beam into two parts A and B. Take A free (Fig. 159*b*). Note that the internal forces acting on CD are represented by V_r, the resisting shear, and M_r, the resisting moment.

Vertical Shear.—The external forces acting on A may be combined into a single force V, the resultant of R_1, P_1, and P_2 (Fig. 159*c*). That is, $V = R_1 - P_1 - P_2$ equals the sum of the vertical forces acting to the left of the section CD. In like

* Since the conditions of Art. 112 are assumed to hold, the internal or resisting shear V_r acting on a section lies in the plane of symmetry containing the loads and reactions.

manner, if B is taken free, $V = R_2 - P_3$ equals the sum of the vertical forces acting to the right of the section CD.

It will be convenient to have a name for V, the resultant of the vertical forces acting to one side of a section of a beam. It is due to V that part A tends to slide on part B along the surface CD. Accordingly, V is called the external or vertical shear for (or at) the section CD. Hence *the external or vertical shear* for (or at) a *section is the resultant of all the vertical forces* (loads and reactions) *that act on the part of the beam that lies to one side of the section* (to the left of the section; or to the right of the section).

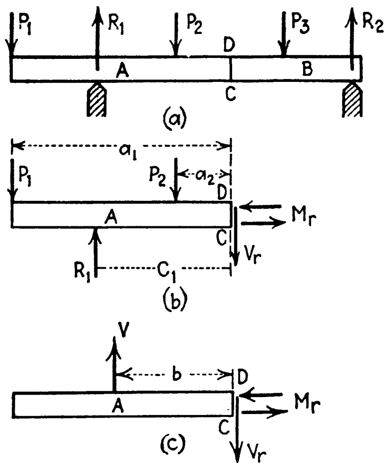


FIG. 159.

Since equilibrium exists and since V and V_r are the only vertical forces acting on part A (Fig. 159*c*), V and V_r must be equal in magnitude but opposite in direction. That is, if V and V_r are assumed to act in opposite directions, $\Sigma F_y = 0$ gives

$$V - V_r = 0 \quad \text{or} \quad V_r = V.$$

In general, *the internal or resisting shear V_r on a section and the external or vertical shear V for that section are equal in magnitude but opposite in direction.* If V acts up (if $R_1 > P_1 + P_2$, Fig. 159), V_r acts down; if V acts down, V_r acts up.

Bending Moment.—Referring to Fig. 159*c*, note that V and V_r, being equal in magnitude but opposite in direction, form a couple VV_r. This couple tends to rotate the beam about the section CD and hence tends to bend the beam. The moment of this couple is called the external or *bending moment* at (or for) the section CD and will be designated by M. That is, by taking the center of moments on the section CD (in order to eliminate V_r),

$$M = Vb.$$

In Theoretical Mechanics it is shown that the moment of the resultant of a system of forces with respect to a point equals

the summation of moments (with respect to that point) of the forces of the system. Hence (Figs. 159b and 159c)

$$M = Vb = R_1c_1 - P_1a_1 - P_2a_2.$$

This leads to the following definition of the bending moment:

The bending moment at (or for) a section of a beam is the algebraic sum of the moments (with respect to that section) of the external forces that lie to one side of that section.

Since equilibrium exists, the external moment M and the internal moment M_r must be equal in magnitude but opposite in direction (Fig. 159c). That is, putting Σ moments $= 0$,

$$M - M_r = 0 \quad \text{or} \quad M_r = M.$$

In general, *the internal* or *resisting moment* M_r *at a section and the external or bending moment* M *at that section are equal in magnitude but opposite in direction.*

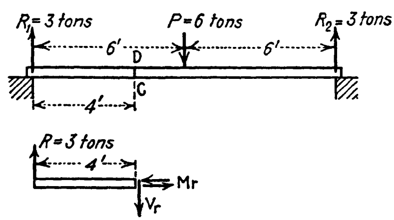


FIG. 160.

Note.—It is important to remember that the internal forces acting on a section hold the external forces acting to one side of that section in equilibrium. That is, the internal forces resist the external forces and therefore are called *resisting forces.*

Example. Fig. 160.—It is required to find the resisting shear and the resisting moment at a section 4 ft. from the left end of the beam.

The reactions are $R_1 = 3$ tons and $R_2 = 3$ tons.

1. Considering forces to the left of the section,

$V = 3$ tons (up);
$M = 3 \times 4 = 12$ ft.-tons (clockwise).

Therefore

$V_r = 3$ tons (down) and $M_r = 12$ ft.-tons (counterclockwise).

2. Considering forces to the right of the section,

$V = 3 - 6 = -3$; or $V = 3$ tons (down).
$M = 3 \times 8 - 6 \times 2 = 12$ ft.-tons (counterclockwise).

Therefore

$$V_r = 3 \text{ tons (up), and } M_r = 12 \text{ ft.-tons (clockwise).}$$

Note.—In general, the resisting shear V_r and the resisting moment M_r at a section are determined as soon as the vertical shear V and the bending moment M at that section are determined. The remaining articles of this chapter are concerned mainly with the vertical shear and the bending moment at various sections of beams.

120. Sign of Bending Moment.—To be consistent, it is necessary to agree upon the algebraic sign to be given a bending

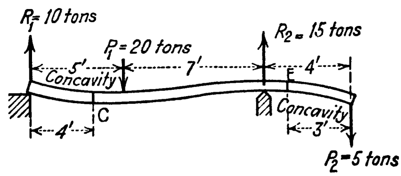


FIG. 161.

moment. A bending moment at a section will be considered positive (+) if at that section the concavity of the beam is above the beam, and negative (−) if the concavity is below the beam. Thus (Fig. 161) the bending moment at C is positive (concavity above), and at E it is negative (concavity below). It should be noted that a positive bending moment produces compression in the upper fibers and tension in the lower, and that a negative bending moment produces tension in the upper fibers and compression in the lower. So the sign of a bending moment will be consistent if the following rule is adopted:

Rule of Sign.—If the forces to the left of the section are used, take clockwise as positive; if the forces to the right are used,

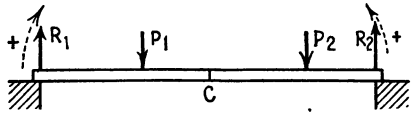


FIG. 162.

take counterclockwise as positive. In Fig. 162, if the forces to the left of the section C are used to find M_C (the bending moment at C), consider the moment of R_1 as positive and the moment of P_1 as negative; if the forces to the right of C are used, consider the moment of R_2 as positive and that of P_2 as negative.

Sometimes it is more convenient to state the rule as follows: *Whether the forces to the left of a section or the forces to the right of the section are used, always take the moment of an upward force as positive (+) and the moment of a downward force as negative (−).*

Illustration. Fig. 161.—The bending moment at the section C is, using forces to the left of the section,

$$M_C = 10 \times 4 = +40 \text{ ft.-tons.}$$

Using forces to the right of the section,

$$M_C = 15 \times 8 - 20 \times 1 - 5 \times 12 = 40 \text{ ft.-tons.}$$

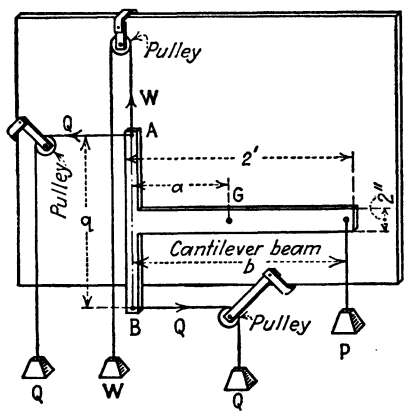


FIG. 163.

The bending moment at E is, using forces to the left of the section,

$$M_E = 10 \times 13 - 20 \times 8 + 15 \times 1 = -15 \text{ ft.-tons.}$$

Using forces to the right of the section,

$$M_E = -(5 \times 3) = -15 \text{ ft.-tons.}$$

121.—In the foregoing articles, it was shown that, if the part of a beam to one side of a section is considered, the internal forces acting on that section may be replaced by a resisting shear $\bar{V}_r$ and a resisting moment M_r (Fig. 158). It was also shown that V_r is equal in magnitude but opposite in sign to the sum of the vertical forces acting on the part of the beam under consideration, and that M_r is equal in magnitude but opposite in sign to the sum of the moments of these external forces with respect to the given section. These conclusions are easily verified experimentally in a particular beam.

Take a board (say 2 ft. by 2 in. by 1 in.). To one end nail a narrow strip AB as shown (Fig. 163). At A and B (say 12 in. apart) make small holes

sufficiently large for a thin strong cord to go through. In front of a large drawing board, set up the cantilever beam as shown. Let G equal the weight of beam, the center of gravity of the beam being a distance a from the line AB. (The center of gravity may be found experimentally by balancing.) Hang a weight (P) as shown, a distance b from AB. Now adjust the weights W and Q until the beam hangs freely in a horizontal position. It will be found that $W = G + P$.

That is,

$$V_r = V.$$

Also

$$Qq = Pb + Ga.$$

That is,

$$M_r = M.$$

PROBLEMS

93. A simple beam of length L rests on end supports and carries a central concentrated load P. Determine the reactions of the supports. Find the expression for the moment under the load. *Ans.* Moment $= PL/4$.

94. If in Problem 93 the load is moved to a point $L/10$ from the center, find the reactions; also the moment under the load.

Ans. Reactions, $0.4P$ and $0.6P$; $M = 0.24PL$.

Compare results with those of Problem 93.

95. A simple beam on end supports is 16 ft. long. It carries a concentrated load of 6 tons 4 ft. from the left end, and a concentrated load of 4 tons 6 ft. from the right end. It is required to find the reactions at the supports, and the vertical shear and the bending moment at a section 8 ft. from the left end. *Ans.* 6 tons, 4 tons; 0; 24 ft.-tons.

96. Solve Problem 95 if, in addition to the loads there given, the beam carries a load of 12 tons uniformly distributed over its whole length.

Ans. 12 tons, 10 tons; 0; 48 ft.-tons.

97. A beam 16 ft. long is supported at the right end and at a point 4 ft. from the left end. At the left end a concentrated load of 1 ton acts. Beginning at the right support and extending to the left over a length of 8 ft. there is a uniformly distributed load of 8 tons. Required to find (*a*) reactions; (*b*) shear at a section just to the left of the left support, just to the right of the left support; (*c*) bending moment at a section 6 ft. from the left support; 4 ft. from the left support.

Ans. (*a*) 4 tons, 5 tons; (*b*) 1 ton, 3 tons; (*c*) 12 ft.-tons, 8 ft.-tons.

98. A simple beam of length L rests on end supports. Beginning at the left support and extending over two-thirds of the length of the beam, there is a uniformly distributed load W (total). Find the reactions. Find also the moment at a section $\frac{1}{2}L$ from the left support.

Ans. $\frac{2}{3}W$, $\frac{1}{3}W$; $\frac{7}{48}WL$.

SHEAR AND MOMENT DIAGRAMS

122. The shear diagram for a beam is a figure whose ordinates measured from a base line represent to some convenient scale the vertical shears for the corresponding sections in the beam.

For instance, Fig. 164*b* is the shear diagram for the beam shown in Fig. 164*a*. That is, an ordinate such as DD' represents to some scale the algebraic sum of all the external forces (loads and reactions) that lie to the left of the corresponding point D in the beam. In determining the vertical shear at a section, the

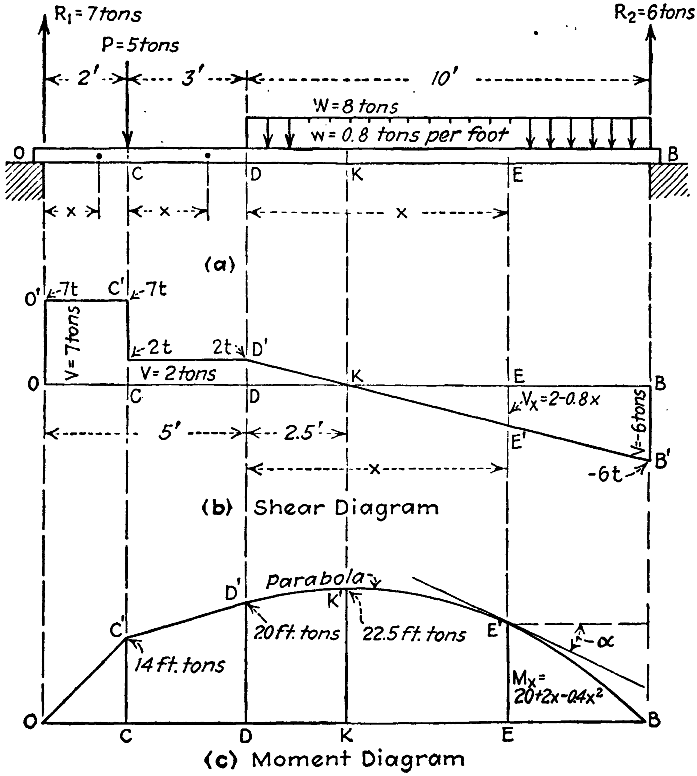


FIG. 164.

forces that lie to the left of the section are used as a rule. Thus the vertical shear at the section D is

$$V_D = 7 - 5 = +2 \text{ tons.}$$

The shear diagram often plays a very important part in the solution of beam problems. The construction of the shear diagram, therefore, should be clearly understood.

Consider the beam shown in Fig. 164*a*. The reactions are determined in the usual way; $R_1 = 7$ tons, and $R_2 = 6$ tons. At any point in the beam between O and C, the vertical shear is

$$V_{OC} = R_1 = 7 \text{ tons.}$$

Hence the shear diagram between O and C [*i.e.*, $O'C'$ (Fig. 164 *b*)] is a horizontal straight line 7 units above the base line OB.

At any point in the beam between C and D, the shear is

$$V_{CD} = R_1 - P = 7 - 5 = 2 \text{ tons.}$$

Hence between C and D the shear diagram is a horizontal straight line 2 units above the base line OB.

To find the shape of the shear diagram between D and B it will be convenient to determine the *shear equation* for the part DB of the beam. Let x equal the distance of the section E from some convenient point (say D) in the beam, and let V_x equal the vertical shear at the section E. Between D and B the load is uniformly distributed at the rate of $w = 0.8$ ton per running foot. Therefore the total load between D and E is $wx = (0.8)x$. The vertical shear at E is

$$V_x = R_1 - P - wx = 7 - 5 - 0.8x.$$

Or

$$V_x = 2 - 0.8x. \tag{a}$$

Equation (a) is the equation of a straight line with the point D as the origin. [If y is put for V_x, Eq. (a) becomes $y = 2 - 0.8x$, the equation of a straight line.] Hence, between D and B, the shear diagram $D'B'$ is a sloping straight line, the shear decreasing at the constant rate of w ($= 0.8$) ton per running foot.

By putting $x = 0$ in Eq. (a), $V_D = 2$ tons as found before; by putting $x = 10$, $V_B = -6$ tons. Hence the shear diagram for the part DB is conveniently constructed by laying off the ordinates DD' and BB' and then joining D' and B' by the straight line $D'B'$.

Note.—An examination of the shear diagram of Fig. 164*b* leads to the following useful propositions:

a. Neglecting the weight of the beam, the shear diagram is a horizontal straight line for any part of the beam that sustains no external forces. (Between O and C, or between C and D, the shear diagram is a horizontal straight line.)

b. Under a concentrated force (load or reaction), the shear abruptly changes, the change in the shear being equal to the force. Thus under the load P the shear diagram abruptly drops 5 units.

c. Under a uniformly distributed load, the shear diagram is a sloping straight line.

123. The moment diagram for a beam is a figure whose ordinates measured from a base line represent the bending

moments at the corresponding sections in the beam. When determining the bending moment at a section, the rule of sign should be observed. Always consider the bending moment of an upward force as positive and that of a downward force as negative.

Consider the beam of Fig. 164*a*. Between O and C, at a distance x from O, the bending moment is

$$M_x = R_1 x = 7x.$$

This is the equation of a straight line, OC' in Fig. 164*c*, with the point O as origin. By putting $x = 2$ as one of the values of x, $M_C = 14$ ft.-tons. Hence the ordinate at C should be drawn up from OB to represent 14 ft.-tons.

Between C and D, at a distance x from C,

$$M_x = R_1(2 + x) - Px = 7(2 + x) - 5x = 14 + 2x.$$

This is the equation of a straight line with C as origin. By putting $x = 3$, $M_D = 20$ ft.-tons.

To determine the moment diagram under the uniformly distributed load, consider a section E distant x from D. wx equals the resultant of the distributed load between D and E (Fig. 164*a*). $x/2$ equals the distance of this resultant from E. The moment of the distributed load between D and E with respect to the section E is

$$wx \times \frac{x}{2} = \frac{wx^2}{2}.$$

The sum of the moments, with respect to E, of all the external forces to the left of E is therefore (E being a distance x from D)

$$M_x = R_1(5 + x) - P(3 + x) - \frac{wx^2}{2}$$

$$= 7(5 + x) - 5(3 + x) - \frac{0.8x^2}{2}.$$

Simplifying,

$$M_x = 20 + 2x - 0.4x^2. \qquad (a)$$

Equation (*a*) is the equation of a parabola $D'K'E'B$ (Fig. 164*c*) with D as the origin. Hence the moment diagram under the uniformly distributed load is a parabola.*

* If y be put for M_x the equation takes the form $y = 20 + 2x - 0.4x^2$. Since this equation is of the first degree in y and of the second degree in x, the equation is that of a parabola. Moreover, it can be shown that its axis of symmetry is vertical.

Note.—An examination of the moment diagram of Fig. 164*c* leads to the following useful propositions:

a'. If the weight of the beam is neglected, the moment diagram is a straight line (usually sloping), for any part of the beam that sustains no external forces. (OC' and $C'D'$ are straight lines.)

b'. Under a concentrated force (load or reaction), the slope of the moment diagram makes an abrupt change. Thus at C' the slope changes from that of OC' to that of $C'D'$. Expressed in another way, the moment diagram is a smooth curve except under concentrated forces.

c'. Under a uniformly distributed load (acting downward) the moment diagram is a parabola *with the concavity below.*

124. Relation between Shear and Moment.—An important theorem will now be developed. Let V equal the vertical shear at a section C in a beam loaded and supported in any manner.

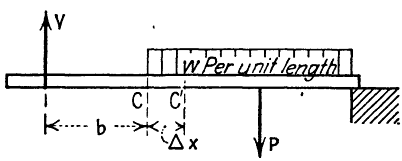


FIG. 165.

The vertical shear is defined as the resultant of all the external vertical forces (loads and reactions) acting (say) to the left of a section. Hence the vertical forces to the left of the section C may be replaced by their resultant V, as is shown in Fig. 165. Assume that V acts at a distance b from the section C. The bending moment at C is therefore

$$M = Vb. \tag{a}$$

Let C' be a section to the right of C and distant Δx from C. As we go from C to C', the bending moment increases by an amount ΔM. Hence the bending moment at C' is

$$M' = M + \Delta M = V(b + \Delta x) - \frac{w(\Delta x)^2}{2}, \tag{b}$$

the last term being the moment with respect to the section C' of the distributed load (if any) lying between C and C'.

Subtracting Eq. (*a*) from Eq. (*b*), member from member,

$$\Delta M = V\Delta x - \frac{w(\Delta x)^2}{2}.$$

Dividing through by Δx,

$$\frac{\Delta M}{\Delta x} = V - \frac{w\Delta x}{2}. \tag{c}$$

Assume now that C' approaches coincidence with C. That is, in the limit as $\Delta x \doteq 0$, Eq. (c) becomes

$$\frac{dM}{dx} = V. \tag{1}$$

Expressed in words: *the value of dM/dx for any given point in the beam equals the vertical shear at that point.*

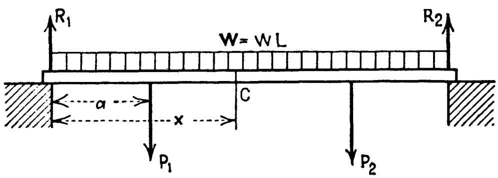


FIG. 166.

Illustration.—Consider the particular case given in Fig. 166. The vertical shear at C is

$$V = R_1 - P_1 - wx. \tag{d}$$

The bending moment at C is

$$M = R_1x - P_1(x - a) - \frac{wx^2}{2}.$$

Differentiating with respect to x,

$$\frac{dM}{dx} = R_1 - P_1 - wx. \tag{e}$$

Comparing Eqs. (d) and (e), it is seen that $dM/dx = V$.

Note.—Equation 1, $dM/dx = V$, has an interesting geometric meaning.

Let A' be a point on a moment curve (Fig. 167). To find the slope of the curve at A' proceed as follows: Let B' be a near point on the curve. Since the ordinate $AA' = M$ equals the bending moment at the corresponding point A in the beam, the ordinate $BB' = M + \Delta M$. Consider the triangle $A'B'C'$. The slope of the secant $A'B'$ is $\Delta M/\Delta x = \tan \theta$. In the limit, as B' approaches coincidence with A', *i.e.*, as $\Delta x \doteq 0$, the secant $B'A'$ approaches coincidence with the geometric tangent at A'.

Hence in the limit as $\Delta x \doteq 0$ we have

$$\frac{dM}{dx} = \text{slope of curve at } A' = \tan \alpha.$$

The equation $dM/dx = V$ may be interpreted therefore as meaning that *the slope of the moment curve at any point of that curve is represented by the ordinate to the corresponding point in the shear curve.*

Illustration.—With reference to Fig. 164*c*, the slope of the moment curve at E' (*i.e.*, $\tan \alpha$) is represented by the ordinate EE' of the shear diagram (Fig. 164*b*).

125. Shear Area. Area of Shear Diagram.—With reference to Fig. 166, the shear diagram for particular values of P_1, P_2,

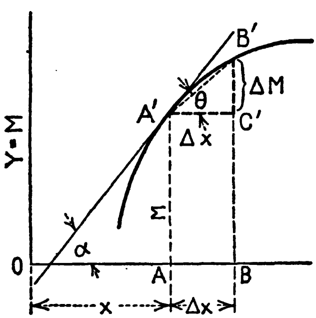


FIG. 167.

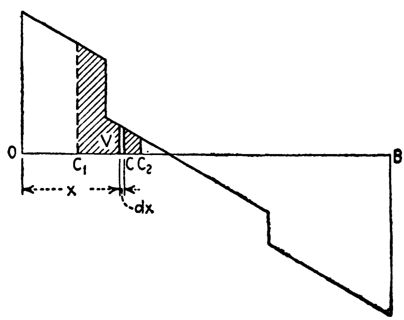


FIG. 168.

and W will take the form shown in Fig. 168. If V equals the ordinate at C (equals vertical shear at the corresponding point in the beam), then $dM/dx = V$ (Art. 124). Write this equation in the form

$$dM = Vdx \tag{a}$$

and integrate between $x = x_1$ and $x = x_2$; *i.e.*, integrate between the two points C_1 and C_2. The integration of Eq. (*a*) gives

$$M_2 - M_1 = \int_{x_1}^{x_2} Vdx = \text{area of shear diagram between } C_1 \text{ and } C_2.$$

Eq. (*b*).

Hence, if C_1 and C_2 are two points in a beam, the increase (or the decrease) in the bending moment, going from C_1 to C_2, equals the area between the corresponding ordinates in the shear diagram. This theorem may be called the *shear area theorem.*

If the shear diagram is constructed from left to right, as is generally done, the integration in Eq. (*b*) should be performed from left to right in order to be consistent with the convention as to the sign of the bending moment (Art. 120).

Important Note.—It is preferable to find the bending moment at a section in a beam by taking the summation of the moments of the external forces acting to one side of the section, always taking the moment of an upward force as positive (+).

Illustration.—Referring to Fig. 164*a*, find the bending moments at C, D, and K from the shear area. $M_O = 0$;

$$M_C = 7 \times 2 = 14 \text{ ft.-tons};$$
$$M_D = 7 \times 2 + 2 \times 3 = 20 \text{ ft.-tons},$$

and

$$M_K = 7 \times 2 + 2 \times 3 + \frac{2 \times 2.5}{2} = 22.5 \text{ ft.-tons}.$$

126. Sections of Maximum Bending Moment.—A bending moment is said to be a *maximum* if *numerically* (*i.e.*, regardless of sign) this moment is larger than the moment at a near section to either side. In Fig. 169*c*, the moment diagram consists of a parabola $Ok'B'$ and a straight line $B'C$. The diagram has two maximum ordinates—one at k and the other at B. Each of these ordinates is numerically larger than an ordinate near it on either side. Hence the largest moment in the beam is either at K or at B (Fig. 169*a*).

The bending moment in a beam, numerically the largest, will be the *true maximum moment* and will be designated without regard to its sign by M_m. If, in the case illustrated

$$M_B = -15 \text{ ft.-tons and } M_k = 14 \text{ ft.-tons},$$
$$M_m = 15 \text{ ft.-tons}.$$

Referring to the moment diagram (Fig. 169*c*), note that at k' and at B' the slope of the curve changes sign. That is, dM/dx passes through a zero value at k' and at B'. Just to the left of k', dM/dx is positive (+); just to the right of k', it is negative (−). Hence, at k', dM/dx passes through the zero value. Since $dM/dx = V$, the shear passes through zero value at the corresponding point K in the beam. This means that the shear diagram must cross the base line at k (Fig. 169*b*). In like manner, just to the left of B' (Fig. 169*c*) dM/dx is negative; just to the right of B' it is positive. That is, the slope of the moment diagram

abruptly changes sign at B', and therefore the shear abruptly passes through zero value at the corresponding point in the beam. Hence the shear diagram must cross the base line at B. This leads to the following very important theorem, true regardless of the type of loading:

At sections where the moment is numerically a maximum, the shear passes through the zero value.

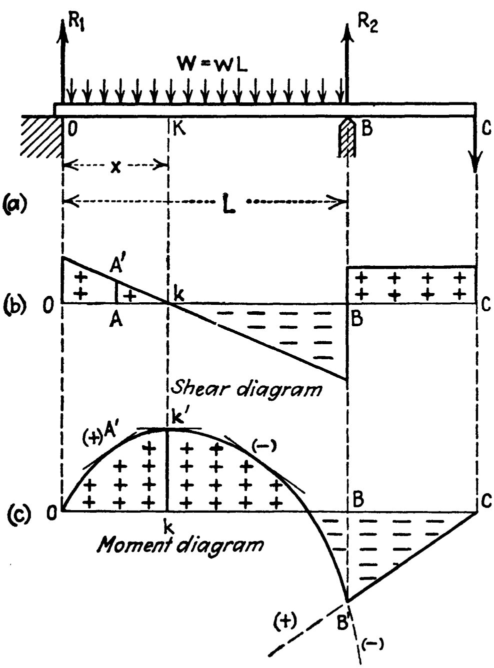


FIG. 169.

This suggests the following procedure for the determination of the location and amount of the maximum moment M_m in a beam:

Construct the shear diagram and locate the points where the shear diagram crosses the base line. Calculate the bending moment at each corresponding section in the beam and compare results to select the largest bending moment M_m.

Illustration. Fig. 169*b*.—Having located the points K and B from the shear diagram, calculate M_K and M_B, the larger in numerical value being M_m.

Note.—By referring to Fig. 169*c*, it is seen that at k' the curve is smooth and the tangent is horizontal. That is, at k'

$$\frac{dM}{dx} = 0.$$

The ordinate kk' is therefore a "calculus maximum." At B', however, the curve is not smooth and dM/dx does not equal zero but abruptly passes through zero, as is indicated in the shear diagram.

127. Location of Sections Where the Shear Passes through Zero.—It will be shown in the next chapter that, if a prismatic beam (a beam of constant cross-section) fails in bending, it will fail at the section where the bending moment is a maximum. The section of maximum bending moment is therefore the *critical* or *dangerous section* in a prismatic beam (not necessarily in other shapes of beams).

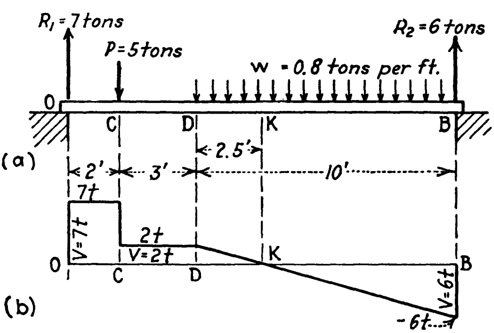


FIG. 170.

If a prismatic beam is to be selected to carry a given loading, or if a load is to be found that a given beam can carry, the maximum bending moment must be computed. To do so, it is first of all necessary to locate the section where the bending moment is a maximum. This is conveniently done by locating the section or sections where the shear passes through the zero value.

Referring to the shear diagram of Fig. 169*b*, note that the point B is readily located. That is, B is directly under the reaction R_2. The point k, however, cannot be determined by inspection. To locate k proceed as follows: Let x equal distance of k from some convenient point, as O. Now V_k, the shear at k, must be zero. That is,

$$V_k = R_1 - wx = 0.$$

Therefore

$$x = \frac{R_1}{w}.$$

Illustration.—Figure 170 in part is a repetition of Fig. 164. It is required to locate the section K in the beam where the

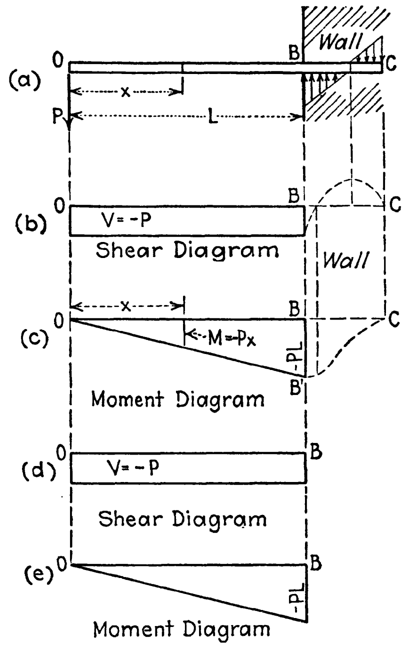


FIG. 171.

shear passes through the zero value. If x equals distance of K from the point D,

$$V_K = 7 - 5 - 0.8x = 0.$$

Solving for x,

$$x = 2.5 \text{ ft.}$$

Therefore, the moment will be a maximum 2.5 ft. to the right of D.

Note.—Since the vertical shear at D is known, having been found when constructing the shear diagram, the shear at K may be written directly as $V_K = V - wx = 2 - 0.8x$.

128. Cantilever Beam.—For the cantilever beam shown in Fig. 171a, the shear at any section between O and B is $V = -P$. The shear diagram between O and B is therefore a straight

horizontal line P units below the base line OB (Fig. 171*b*). The moment at a section between O and B is $M = -Px$. Hence the moment diagram is a sloping straight line OB' with

$$BB' = -PL \text{ (Fig. 171}c\text{).}$$

It is not known just how the pressure of the wall is distributed over the part BC—the part built into the wall. If it is assumed that the pressure is distributed as shown in Fig. 171*a*, it can be shown that the shear and moment diagrams for the part BC have somewhat the forms shown by the dotted lines in Figs. 171*b* and 171*c*, respectively.

It is sufficient, as far as the beam is concerned, to construct the shear and the moment diagram only for that part of the cantilever which projects from the wall. Accordingly, the shear and the moment diagram for the cantilever are given by Figs. 171*d* and 171*e*, respectively.

129. Shear and Moment under a Concentrated Load.—A concentrated load is a load distributed over so small a length of the beam that *for practical purposes* this load may be thought of as acting at a point (the point of application of the resultant of the load). In reality, however, all loads are distributed.

Consider the beam shown in Fig. 172*a*. Assume that the load is distributed uniformly over the length Δx. The shear diagram under the uniformly distributed load P is a sloping straight line (Fig. 172*b*), and the moment diagram is a parabola (Fig. 172*c*). If KK' is the maximum ordinate in the moment diagram, the tangent at K' is horizontal. Hence, at K', $dM/dx = 0$; *i.e.*, KK' is a calculus maximum.

For practical purposes, however, it will be found much more convenient to represent loads distributed over a small length of the beam as a load acting at a point, *i.e.*, as a concentrated load, and to draw the shear and moment diagrams on this assumption (Fig. 173).

Note.—When a load is represented as a concentrated load and the shear diagram drawn accordingly, it is meaningless to speak of the shear under the concentrated load. In Fig. 173, for instance, under the concentrated load P the shear abruptly changes from KK' to KK''. It is necessary, therefore, to speak of the shear *just to the left*, or *just to the right* of the load. In Fig. 173*b*, KK' is the shear just to the left of K, and KK'' is the shear just to the right of K. In this discussion, a reaction may be the load considered.

130. Examples.—When there is but one maximum moment in a beam, this moment is the largest moment. When there are two or more maximum moments, each maximum may be called a *local maximum*, and the largest of these local values (the largest moment in the beam) will be called the *maximum moment.* A similar statement holds for the maximum vertical shear in a beam.

In each of the following examples it is required to construct the shear and moment diagrams, and to determine the maximum shear V_m and the maximum moment M_m. Note particularly the method of procedure used in these examples. This method is quite a general one and therefore should be carefully studied and clearly understood.

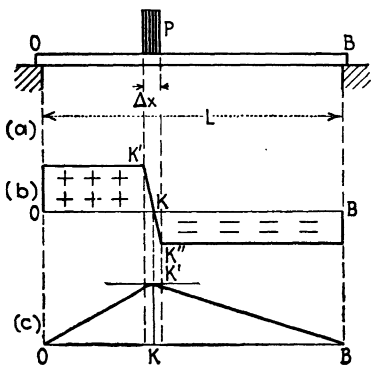


FIG. 172.

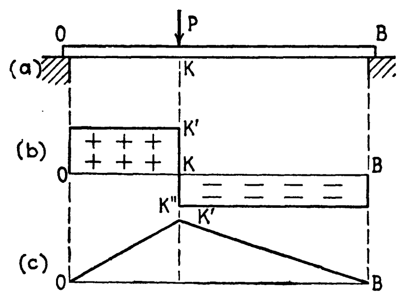


FIG. 173.

The propositions given in the *notes* of Arts. 122 and 123 will be used. It is advisable to reread these notes before beginning the study of the examples that will now be given.

Example I. Fig. 174.—Neglect the weight of the beam.

1. *Reactions.*—These are found in the usual way.

$$\Sigma M_O = 0;\ R_2 \times 15 - 3 \times 12 - 5 \times 7 - 3 \times 3 = 0.$$

Or

$$R_2 = 5.33 \text{ tons.}$$
$$\Sigma M_B = 0;\ R_1 \times 15 - 3 \times 12 - 5 \times 8 - 3 \times 3 = 0.$$

Or

$$R_1 = 5.67 \text{ tons.}$$

2. *Shear Diagram.*—The shear at any section between O and C is

$$V_{OC} = 5.67 \text{ tons.}$$

At C, the shear drops 3 tons.
Therefore

$$V_{CD} = 5.67 - 3 = 2.67 \text{ tons.}$$

At D, the shear drops 5 tons.
Therefore

$$V_{DE} = 2.67 - 5 = -2.33 \text{ tons.}$$

At E, the shear drops 3 tons.
Therefore

$$V_{EB} = -2.33 - 3 = -5.33 \text{ tons,}$$

thus checking against the reaction R_2.

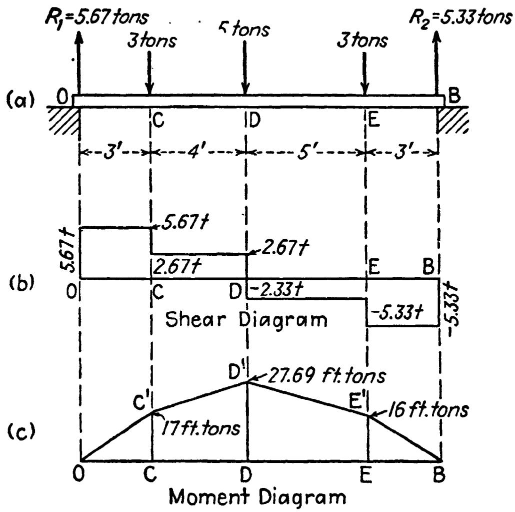


FIG. 174.

Maximum Shear.—From the diagram it is seen that V is a maximum between O and C. That is, $V_m = 5.67$ tons.

3. *Maximum Moment.*—Since the shear crosses the base line at D, the maximum moment occurs at D. Hence (Fig. 174*a*)

$$M_D = 5.67 \times 7 - 3 \times 4 = 27.69 \text{ ft.-tons} = M_m.$$

4. *Moment Diagram.*—Calculate the moment at every section where there is a concentrated load, *i.e.*, at C, D, and E. The moment at D happens to be the maximum moment and has been found.

$$M_C = 5.67 \times 3 = 17.0 \text{ ft.-tons.}$$

$$M_D = 27.67 \text{ ft.-tons.}$$
$$M_E = 5.33 \times 3 = 16.0 \text{ ft.-tons.*}$$

Plot these moments (Fig. 174*c*) and draw the straight lines OC', $C'D'$, $D'E'$, and $E'B$ (Note a', Art. 123).

Example II. Fig. 175.—Neglect the weight of the beam.

1. *Reactions:* $R_1 = 5.89$ tons. $R_2 = 6.61$ tons.

2. *Shear Diagram.*—Just to the right of O, $V = R_1 = 5.89$ tons. (Plot to some scale.)

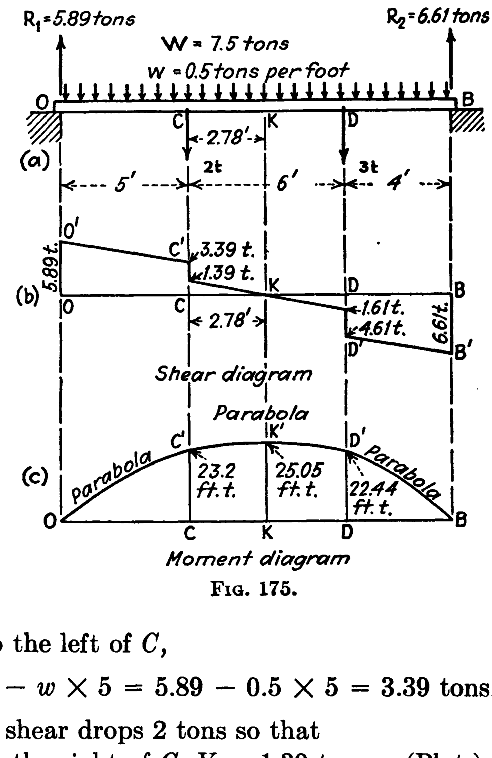


FIG. 175.

Just to the left of C,

$$V = R_1 - w \times 5 = 5.89 - 0.5 \times 5 = 3.39 \text{ tons.} \quad \text{(Plot.)}$$

At C the shear drops 2 tons so that

Just to the right of C, $V = 1.39$ tons. (Plot.)

Just to the left of D, $V = 1.39 - 0.5 \times 6 = -1.61$. (Plot.)

At D the shear drops 3 tons so that

Just to the right of D, $V = -4.61$ tons. (Plot.)

* In finding the moment at a section, either the moments of the forces acting on the portion of the beam extending to the left of the section or the moments of the forces acting on the portion of the beam extending to the right of the section may be used. In either case, the moment of a force acting upward is considered positive (Art. 120).

Just to the left of B, $V = -4.61 - 0.5 \times 4 = -6.61$, thus checking against the reaction R_2.

Maximum shear occurs just to the left of B and is (without regard to sign) $V_m = 6.61$ tons.

3. *Maximum Moment.*—The shear diagram crosses the base line at K.

To locate K, use the principle that, at K, $V = 0$ (Note, Art. 127).

$$V_K = 1.39 - 0.5x = 0.$$

Therefore

$$x = 2.78' \text{ from } C.$$

The moment at K is (Fig. 175*a*)

$$M_K = 5.89 \times 7.78 - 2 \times 2.78 - \frac{0.5(7.78)^2}{2} = 25.05 \text{ ft.-tons}$$
$$= M_m.$$

4. *Moment Diagram.*—Calculate the moment at each section where there is a concentrated load, *i.e.*, at C and at D.

$$M_C = 5.89 \times 5 - \frac{0.5(5)^2}{2} = 23.20 \text{ ft.-tons.}$$

$$M_D = 6.61 \times 4 - \frac{0.5(4)^2}{2} = 22.44 \text{ ft.-tons.}$$

Plot the moments M_C and M_D. Also plot $M_K = M_m$. Draw the curves OC', $C'D'$, and $D'B$. These curves represent different parabolas (Note c', Art. 123).

Note.—As a rule, it will suffice to approximate the shape of these parabolas. In drawing the curve representing the parabola $C'D'$, it should be remembered that K' is the highest point in this curve. Since in this case the moment curve at K' is smooth, the tangent to the curve at K' should be horizontal. Note also that at C' and D' the moment curve is not smooth (Note b', Art. 123); *i.e.*, at C' and D' the slope of the curve *abruptly* changes.

Example III. Fig. 176.—The cantilever beam weighs 40 lb. per running foot and carries a concentrated load of 600 lb. as shown. Treat the weight of the beam as a uniformly distributed load. Verify all results given.

1. *Reactions.*—There are no reactions to the left of B.
2. *Shear Diagram.*—Fig. 176*b*: $V_m = 1000$ lb.
3. *Maximum Moment.*—Fig. 176*c*: $M_m = -5000$ ft.-lb.

4. *Moment Diagram.*—

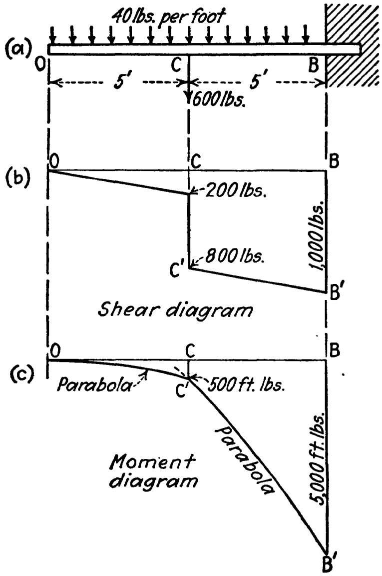


FIG. 176.

$$M_C = -500 \text{ ft.-lb.}$$
$$M_B = -5000 \text{ ft.-lb.}$$

Plot M_C and M_B (Fig. 176*c*). Draw OC' and $C'B'$ to represent parabolas.

Example IV. Fig. 177.—Over-hanging beam. Neglect its own weight. Verify all results given.

1. *Reactions:* $R_1 = 10$ tons. $R_2 = 4$ tons.

2. *Shear Diagram:* $V_{AO} = -3$ tons. At O, the shear jumps up 10 tons. Hence $V_{OC} = +7$ tons. At C, the shear drops 5 tons. Hence $V_{CD} = +2$ tons. At D, the shear begins to decrease at the rate of $w = 0.6$ ton/ft. Hence, just to the left of B, $V_B = 2 - 6 = -4$ tons, thus checking against the reaction R_2.

Maximum shear $= V_m = 7$ tons.

3. *Maximum Moment:* $M_O = -3 \times 5 = -15$ ft.-tons. To locate K (Fig. 177*b*), put $V_K = 0$. $V_K = 2 - 0.6x = 0$. Then $x = 3.33$ ft. from D or 6.67 ft. from B. To find M_K, it is simpler to use forces to the right of K.

$$M_K = 4 \times 6.67 - \frac{0.6(6.67)^2}{2} = 13.33 \text{ ft.-tons.}$$

Hence the largest moment occurs at O and is $M_m = 15$ ft.-tons.

4. *Moment Diagram.* Fig. 177*c*.

$$M_O = -15 \text{ ft.-tons. (Plot below base line.)}$$
$$M_C = -3 \times 8 + 10 \times 3 = 6 \text{ ft.-tons.}$$

Taking forces to the right of D,

$$M_D = 4 \times 10 - 6 \times 5 = 10 \text{ ft.-tons.}$$
$$M_K = 13.33 \text{ ft.-tons as found above.}$$

Draw AO', $O'C'$, and $C'D'$ as straight lines, and $D'K'B$ to represent a parabola with K' as its highest point. Note that at D', there being no concentrated force at D, the moment curve is smooth.

PROBLEMS

99. A beam 20 ft. long rests on end supports, O and B. The beam carries a uniformly distributed load of 12 tons and also a concentrated load of 5 tons at 4 ft. from B. Draw the shear and moment diagrams. Compute the maximum moment and the maximum shear.

Ans. $M_m = 40.8$ ft.-tons; $V_m = 10$ tons.

100. In Problem 99, an additional load of 4 tons is placed 5 ft. from O. Locate and find the maximum moment and the maximum shear.

Ans. $x = 10$ ft. from O; $M_m = 50$ ft.-tons; $V_m = 11$ tons.

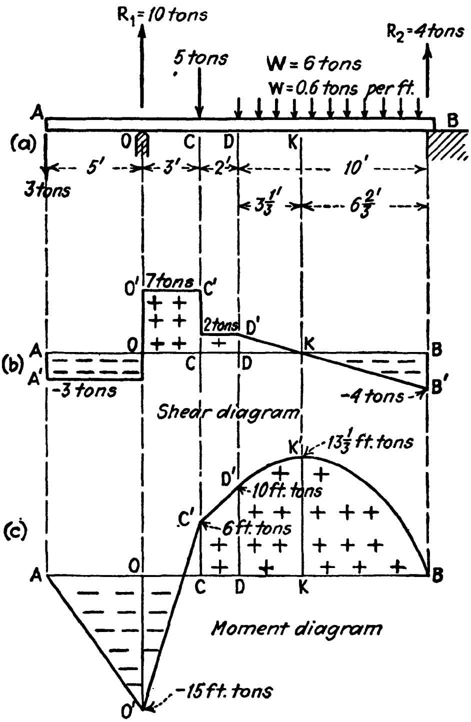


FIG. 177.

101. In Problem 99, move the support O 4 ft. toward B. Solve as before.

Ans. $M_m = 28.8$ ft.-tons; $V_m = 8.25$ tons.

102. In Problem 101, place a load of 3 tons at the outer end of the overhanging part. Solve as in Problem 99.

Ans. $M_m = 25.2$ ft.-tons; $V_m = 7.5$ tons.

103. In Example III (Fig. 176), an additional load of 800 lb. is placed at O. Solve.

131. Uniformly Loaded Beam with Two Symmetrical Supports. Fig. 178.—It is required to locate the position of the

supports so that the maximum moment M_m is less than that for any other position of the supports.

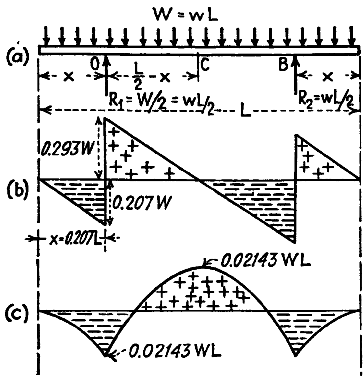


FIG. 178.

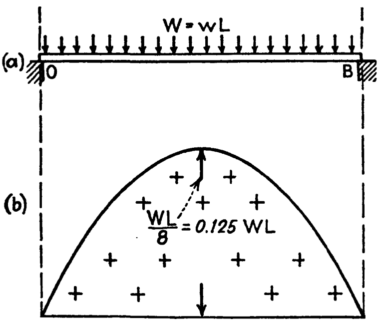


FIG. 179.

If a uniformly loaded beam rests on end supports (Fig. 179), the maximum moment occurs at the center and is

$$M_C = \frac{WL}{8} = 0.125WL. \quad (a)$$

Assume now that the supports are moved in, each by an amount x (Fig. 178). The moment at C is decreased and that at O (and B) is numerically increased. It should be noted that M_O and M_C are of opposite signs.

$$M_O = -\frac{wx^2}{2}.$$

$$M_C = \frac{wL}{2}\left(\frac{L}{2} - x\right) - \frac{wL^2}{8}.$$

Evidently, the supports will be most advantageously placed (as far as moment is concerned) if M_O and M_C are numerically equal. Therefore, put

$$\frac{wx^2}{2} = \frac{wL}{2}\left(\frac{L}{2} - x\right) - \frac{wL^2}{8}$$

and solve for x.

$$x = 0.207L. \quad (b)$$

With this position of the supports, the moments at O, C, and B are all numerically equal.

Therefore

$$M_m = \frac{wx^2}{2} = \frac{w(0.207L)^2}{2} = 0.02143wL^2$$
$$= 0.02143WL. \quad (c)$$

By comparing Eqs. (a) and (c), it is seen that the maximum moment in a uniformly loaded beam with end supports is 5.83 times the moment in the same beam with the supports placed

0.207L in from the ends. Note that 0.207L is a little more than $\frac{1}{5}L$. Hence it is advantageous to place the supports a little more than $L/5$ in from the ends. For a beam 15 ft. long, the maximum moment will be least if the supports are placed 3.11 ft. in from the ends.

Note.—If the moments at O and C are equal in *magnitude* and *sign*, the supports must be together at the middle.

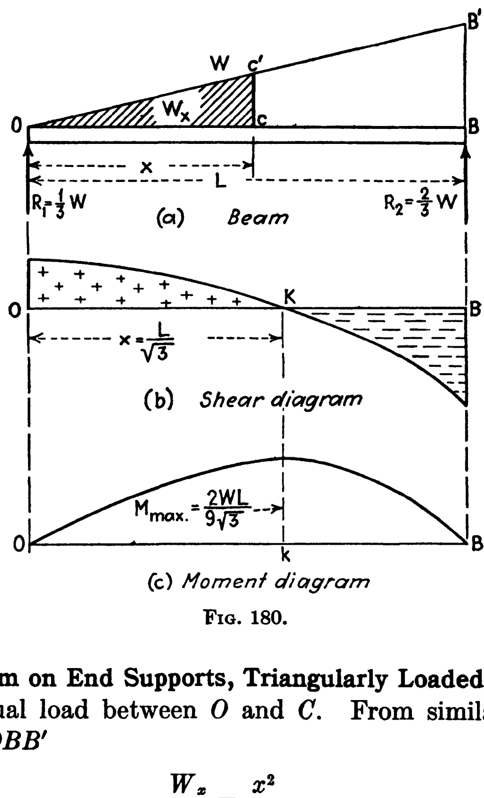


FIG. 180.

132. Beam on End Supports, Triangularly Loaded. Fig. 180. Let W_x equal load between O and C. From similar triangles OCC' and OBB'

$$\frac{W_x}{W} = \frac{x^2}{L^2}.$$

Or

$$W_x = \frac{x^2}{L^2}W. \qquad (a)$$

Reactions:

$$R_2L - W \times \frac{2L}{3} = 0.$$

Therefore

$$R_2 = \frac{2W}{3} \quad \text{and} \quad R_1 = \frac{W}{3}.$$

Shear diagram:

$$V_x = \frac{W}{3} - W_x = \frac{W}{3} - \frac{x^2W}{L^2}. \tag{b}$$

This is the equation of a parabola (Fig. 180*b*).

Moment diagram:

$$M_x = \frac{W}{3}x - \frac{W_x x}{3} = \frac{Wx}{3} - \frac{Wx^3}{3L^2}. \tag{c}$$

This is the equation of a cubic (Fig. 180*c*).

Maximum Moment.—To locate the position of maximum moment, put $V_x = 0$ [Eq. (*b*)], and solve for x.

$$x = \frac{L}{\sqrt{3}} = 0.577L. \tag{d}$$

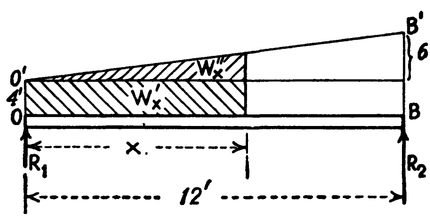


Fig. 181.

Substituting this value of x into Eq. (*c*),

$$M_m = \frac{WL}{3\sqrt{3}} - \frac{WL}{9\sqrt{3}} = \frac{2WL}{9\sqrt{3}} = 0.1282WL. \tag{e}$$

If the weight W were uniformly distributed (rectangular loading), the maximum moment would be $M_m = 0.125WL$. Hence, as far as moment is concerned, rectangular loading is less trying to a beam than triangular loading.

Note.—If the beam is trapezoidally loaded (Fig. 181), it is convenient to resolve the loading into a rectangular and a triangular loading.

Example.—A beam on end supports carries a trapezoidal brick wall (Fig. 181). The wall weighs 125 lb./cu. ft. The lower end of the wall (OO') is 4 ft. high and the upper end (BB') is 10 ft. The beam is 12 ft. long and the wall is 1.5 ft. wide. Assuming

that the brick wall is not counted upon to help support the load, required the value of the maximum moment.

Consider the trapezoid as a combination of a rectangle and a triangle in the side view.

For the rectangular block of wall,

$$W' = 4 \times 12 \times 1.5 \times 125 = 9000 \text{ lb.}$$

For the triangular block,

$$W'' = \frac{6 \times 12}{2} \times 1.5 \times 125 = 6750 \text{ lb.}$$

Reactions:

$\Sigma M_O = 0;$

$$R_2L - \frac{W'L}{2} - \frac{W''2L}{3} = 0.$$

Or

$$R_2 \times 12 - 9000 \times 6 - 6750 \times 8 = 0.$$

$$R_2 = 9000 \text{ lb.,} \qquad \text{and} \qquad R_1 = 6750 \text{ lb.}$$

Shear:

$$V_x = R_1 - \frac{W'x}{L} - \frac{W''x^2}{L^2}.$$

Putting $V_x = 0$,

$$6750 - \frac{9000x}{12} - \frac{6750x^2}{144} = 0.$$

Simplifying,

$$x^2 + 16x - 144 = 0.$$

Solving for x,

$$x = 6.43 \text{ ft.}$$

Maximum moment:

$$M_m = R_1x - \frac{W'x^2}{2L} - \frac{W''x^3}{3L^2}.$$

Or, putting $x = 6.43'$,

$$M_m = 6750 \times 6.43 - \frac{9000(6.43)^2}{2 \times 12} - \frac{6750(6.43)^3}{3 \times 12^2}$$

$$= 23{,}770 \text{ ft.-lb.}$$

PROBLEMS

104. Draw the shear and moment diagrams for the loaded beam shown in Fig. 182. Determine the maximum moment and the maximum shear.

Ans. $M_m = 6$ ft.-tons; $V_m = 2.25$ tons.

105. On the beam of Problem 104, place an additional load of 4 tons uniformly distributed between O and B. Find the maximum moment and the maximum shear. *Ans.* $M_m = 12.07$ ft.-tons; $V_m = 4.25$ tons.

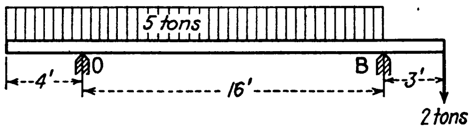


FIG. 182.

106. The beam of Fig. 183 is supported at two points and is loaded as shown. Draw the shear and the moment diagram. Determine the maximum moment and the maximum shear.

Ans. $M_m = 1.32Pa$; $V_m = 1.625P$.

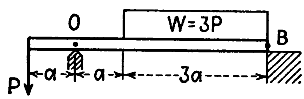


FIG. 183.

107. For a system of coplanar parallel forces, there are only two independent equations of condition for equilibrium. Referring to Fig. 184, take $\Sigma M_O = 0$ and $\Sigma M_B = 0$ and show that by combining these equations algebraically we may derive $\Sigma F = 0$ (see note, Art. 111).

Suggestion.—Between $\Sigma M_O = 0$ and $\Sigma M_B = 0$, eliminate L_2.

108. Referring to Fig. 184, show that, if $\Sigma M_O = 0$ and $\Sigma M_B = 0$, then necessarily $\Sigma M_C = 0$. Consequently, if we put $\Sigma M_O = 0$ and $\Sigma M_B = 0$, $\Sigma M_C = 0$ will not be an independent equation.

Suggestion.—Eliminate R_C between the two equations $\Sigma M_O = 0$ and $\Sigma M_B = 0$.

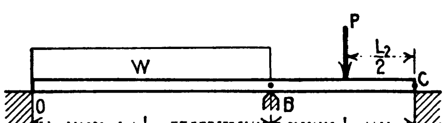


FIG. 184.

109. A beam is triangularly loaded (Fig. 180). Obtain the expression for M_x [see Eq. (*c*), Art. 132], and then use the differential calculus to prove that the position of the section of maximum moment is given by

$$x = \frac{L}{\sqrt{3}}.$$

110. With reference to Fig. 185, find an expression for M_x. Then use the differential calculus to locate the section of maximum moment.

Ans. $x = 9.8$ ft.

111. In Problem 110, determine the value of the maximum moment and of the maximum shear. *Ans.* $M_m = 47.6$ ft.-tons; $V_m = 12.5$ tons.

112. A beam resting on two supports is 30 ft. long. The maximum moment which the beam can carry is 12 ft.-tons. What is the maximum uniformly distributed load that may be applied to the beam if the supports may be placed so as to require the smallest beam, *i.e.*, to produce the least bending moment? *Ans.* 18.65 tons.

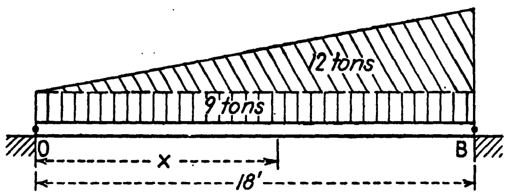


FIG. 185.

113. What is the maximum load that the beam of Problem 112 can carry if the supports are placed at the ends? *Ans.* 3.2 tons.

114. The 12-ft. beam supports the 9000-lb. triangular load. Compute sufficient data and draw the shear and moment diagrams. $V_m = ?$; $M_m = ?$
Ans. $V_m = 4500$; $M_m = 7425$ ft.-lb.

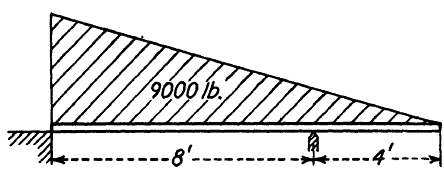


FIG. 186.

115. (Fig. 187) The 20-ft. beam supports the 12-ton triangular load. Compute sufficient data and draw the shear and moment diagrams. $V_m = ?$; $M_m = ?$ *Ans.* $V_m = 9$ tons; $M_m = 24$ ft.-tons.

116. Apply an additional, uniformly distributed load of 12 tons over the 12 ft. of the beam of Problem 115. Solve.

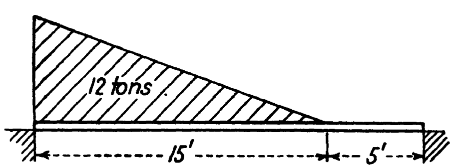


FIG. 187.

117. Apply an additional, concentrated load of 4 tons at a distance of 4 ft. from the right support to the beam of Problem 115. Solve.

CHAPTER VII

STRESSES IN BEAMS

STRESSES DUE TO FLEXURE

133. In Art. 116 it was seen that, if the part to the left (or to the right) of a section of a loaded beam is taken free, the normal stress acting on that section is equivalent to a stress-couple whose moment M_r is called the *resisting moment* on (or at) that section (Fig. 188). It is now desired to develop an expression for M_r such that, if M_r is known, the intensity of the normal stress at any given point in the section may be calculated.

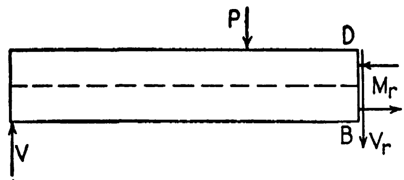


FIG. 188.

Before developing the expression for M_r we shall restate what is meant by the neutral surface and the neutral axis. Figure 189 represents the part of the beam to the left of the section BD so drawn as to expose the section BD to view. The section is represented as rectangular merely for convenience. The plane Ln is the *neutral surface.* Fibers along this surface are neither contracted nor elongated. The intersection of the neutral surface Ln and the section BD, *i.e.*, the line nm, is the *neutral axis* of the section BD. Along this line there are no normal forces acting on the section.

The intensity of the stress on an element of area dA, distant y from the neutral axis nm, will be designated by S_y, and that on an outer element, distant c from the neutral axis, will be designated by S, as is indicated in Fig. 189. The shearing stress acting on an elementary area is not shown in the figure.

Note that an element of area dA may be thought of as the end of a slender horizontal prism, *i.e.*, as the end of a fiber.

134. Additional Assumptions.—In addition to the assumptions made in Art. 112, it will be assumed that

1. The elastic limit is not exceeded.

2. Cross-sections (such as BD, Fig. 189) remain plane surfaces during bending.

3. $E_c = E_t = E$. That is, it will be assumed that the modulus of elasticity for compression may be taken equal to that for tension.

4. Hooke's law may be applied. That is, it will be assumed that the stress in a fiber is directly proportional to the strain (elongation or contraction) of that fiber.

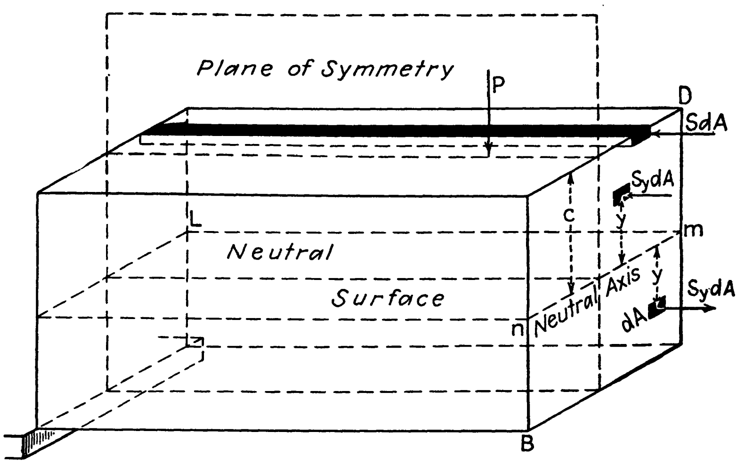


Fig. 189.

5. The beam acts as a single unit. That is, it will be assumed that within the elastic limit the deformation of the beam is one of bending of the beam *as a whole* and not one of twisting, local wrinkling, or buckling of any of its parts. If an I-beam is used, it will be assumed that within the elastic limit there will be no twisting of the beam, wrinkling of the web, buckling of the flange, etc.

Assumptions 2, 3, and 4 are approximations. Experiments show, however, that these assumptions, when applied to materials commonly used for beams, are sufficiently exact for ordinary engineering purposes provided assumption 1 is satisfied—provided the elastic limit is not exceeded. Assumption 5 requires special consideration and will be considered in a later chapter.

Referring to Fig. 189, note that the neutral axis nm is perpendicular to the plane of loading. Obviously, this will always be the case if the beam is symmetrically loaded.*

135. Distribution of the Normal Stress on a Section of a Symmetrically Loaded Beam.—Sections are assumed to remain plane surfaces during bending. Thus (Fig. 190) the straight lines BD and FG, originally parallel and at a distance Δx apart, represent two sections after bending (bending much exaggerated).

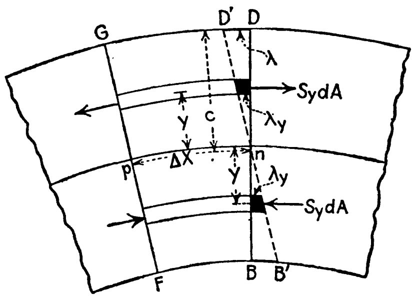


FIG. 190.

Draw the straight line $B'D'$ parallel to FG. If λ_y equals the elongation (or contraction) of a fiber distant y from the neutral surface pn, and λ equals the elongation (or contraction) of an outer fiber distant c from pn, then, from geometry,

$$\frac{\lambda_y}{\lambda} = \frac{y}{c}. \tag{a}$$

Hence *the elongation (or contraction) of a fiber is directly proportional to the distance of this fiber from the neutral surface.*

Let Δx equal the original length of a fiber (all fibers originally of the same length) and let E equal the modulus of elasticity (assumed constant and the same for all fibers within the elastic limit). Treat a fiber as a prism in tension (or compression). Now, in Art. 53, it was shown that, if a prism of length L is subjected to a direct tension (or compression) of intensity S, then within the elastic limit,

$$\text{Elongation of prism} = \frac{SL}{E}.$$

In the present case, the original length of a prism fiber is $L = \Delta x$.

* For unsymmetrical loading, see Art. 154.

Accordingly,

$$\lambda_y = \frac{S_y \Delta x}{E} \qquad \text{and} \qquad \lambda = \frac{S \Delta x}{E}. \tag{b}$$

Since $E_c = E_t = E$ (by assumption), Eq. (*b*) will hold for two fibers on opposite sides of the neutral surface (one in tension and the other in compression).

Dividing the first of the two equations [Eqs. (*b*)] by the second, member by member,

$$\frac{\lambda_y}{\lambda} = \frac{S_y}{S} = \frac{y}{c} \text{ [see Eq. } (a)].$$

Or

$$S_y = \frac{Sy}{c}. \tag{1}$$

Hence *within the elastic limit the intensity of the fiber stress at any point in a section is directly proportional to the distance of this point from the neutral axis of that section.*

Note.—As a rule, a vertical shear also acts on a section of a beam. It will be shown later (Art. 170) that the vertical shear in a beam has no appreciable effect upon the fiber stresses in that beam.

Example.—At a given section of a steel beam the outermost fiber is $c = 6$ in. from the neutral axis. If at this section the stress in the outer fiber is $S = 12{,}000$ lb./sq. in., what is the stress in a fiber 4 in. from the neutral axis?

$$S_y = \frac{y}{c}S \qquad \text{or} \qquad S_4 = \frac{4}{6} \times 12{,}000 = 8000 \text{ lb./sq. in.}$$

136. Stress Figure.—The distribution of the normal stress on a section of a beam is conveniently shown by a stress figure (Fig. 191), *i.e.*, by representing the *intensity* of stress at various distances from the neutral axis *n*. Since the intensity of stress at a point in the section *BD* is directly proportional to the distance of this point from the neutral axis (provided the elastic limit is not exceeded), it follows that the stress figure consists of two similar triangles, one above and one below *n*.

Note.—It is sometimes more convenient to draw the stress figure as shown in Fig. 192.

137. Center of Gravity. Moment of Inertia.—The determination of the intensity of the stress acting at a point on a section of a

beam involves those summations which as integral forms lead to the concepts of the center of gravity and moment of inertia of the section. It is assumed that the reader is familiar with the theory of the center of gravity and of the moment of inertia of plane figures.

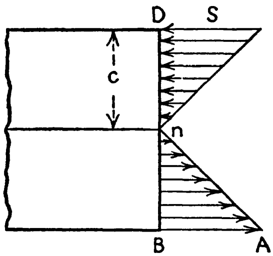


FIG. 191.

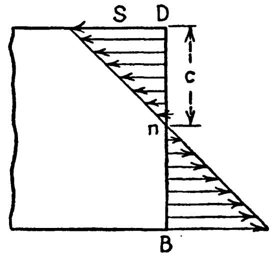


FIG. 192.

Center of Gravity.—Given a plane figure (a section, say, Fig. 193). Take the line X as an axis of reference. If dA is an element of area distant y from the axis X, then the position of the center of gravity of the whole area is determined by the equation

$$A\bar{y} = \int y dA \tag{2}$$

where A = total area of figure.
$\bar{y}$ = distance of its center of gravity from the axis X.

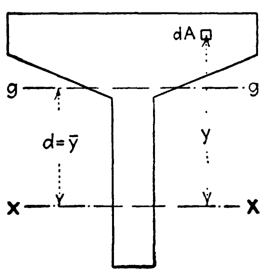


FIG. 193.

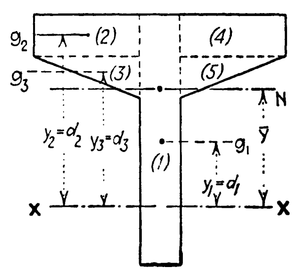


FIG. 194.

Moment of Inertia.—The moment of inertia of the plane figure Fig. 193) with respect to the axis X is defined as

$$I_x = \int y^2 dA. \tag{3}$$

Note.—The moment of inertia frequently is written

$$I = Ak^2 \tag{4}$$

where k is called the *radius of gyration* of the section and is determined from the equation $k^2 = I/A$.

If I_g equals the moment of inertia of the area with respect to its own gravity axis, and d equals the distance of its gravity axis from the axis of reference X (X being parallel to the gravity axis g),

$$I_x = I_g + Ad^2. \tag{5}$$

Equation (5) is called the "parallel axis theorem."

Composite Area.—Frequently an area may be broken up into partial areas (A_1, A_2, A_3, etc.) in such a way that the area and the center of gravity of each partial area may be readily determined (Fig. 194). In such a case, the gravity axis N of the whole figure is located by the equation

$$(A_1 + A_2 + \cdots)\bar{y} = A_1y_1 + A_2y_2 + \cdots \tag{6}$$

where y_1, y_2, etc., are the distances of the centers of gravity of the respective partial areas from the axis of reference X. Equation (6) may be written in the form

$$(\Sigma A)\bar{y} = \Sigma(Ay). \tag{7}$$

In like manner, if I_g equals the moment of inertia of a partial area with respect to its own gravity axis g, A equals the area of this part, and d equals the distance of its center of gravity from the axis of reference X, then

$$I_x = \Sigma(I_g + Ad^2). \tag{8}$$

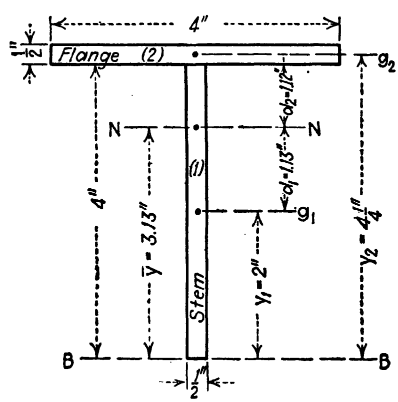


FIG. 195.

Example. Fig. 195.—Taking B as the axis of reference, locate the gravity axis N of the section. Also find I_N. Equation (6)

gives

$$\left(\tfrac{1}{2} \times 4 + 4 \times \tfrac{1}{2}\right)\bar{y} = \left(\tfrac{1}{2} \times 4\right) \times 2 + \left(4 \times \tfrac{1}{2}\right) \times 4\tfrac{1}{4}$$

or

$$4\bar{y} = 12.5$$

Therefore

$$\bar{y} = 3.13 \text{ in.}$$

Take N as the axis of reference and use Eq. (8)*.

Stem:

$$\tfrac{1}{12} \times \tfrac{1}{2} \times 4^3 + \left(\tfrac{1}{2} \times 4\right) \times (3.13 - 2)^2 = 5.23$$

Flange:

$$\tfrac{1}{12} \times 4 \times \left(\tfrac{1}{2}\right)^3 + \left(4 \times \tfrac{1}{2}\right) \times (4.25 - 3.13)^2 = \underline{2.55}$$

Therefore

$$I_N = 7.78 \text{ in.}^4$$

Note.—When a figure is considered as a composite figure, it is convenient to designate the neutral axis (gravity axis) of the entire figure by N. If the figure is considered as a single unit, the neutral axis will be designated as the $g \cdot \cdot \cdot g$ axis.

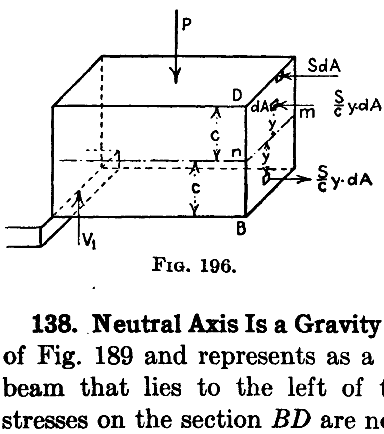


FIG. 196.

Problem 118.—Compute the moment of inertia I_N of the section of an 14-in. 87-lb. WF beam.

Ans. $I = 966.9$ in.^{4}

Note.—In the Appendix, a table will be found giving the properties of WF sections.

138. Neutral Axis Is a Gravity Axis.—Figure 196 is a repetition of Fig. 189 and represents as a free body that part of a loaded beam that lies to the left of the section BD. The shearing stresses on the section BD are not represented.

Let S equal the intensity of stress on an outer element distant c from the neutral axis nm, and let S_y equal the intensity of stress on an element distant y from nm. By assuming now that the assumptions of Art. 134 are satisfied and that therefore the intensity of stress on an element of area is directly proportional to the distance of this area from the neutral axis, it follows that

* The moment of inertia of a rectangle of width b and height h with respect to its gravity axis (parallel to b) is $I_g = \frac{1}{12}bh^3$. Note that I_g is of the fourth dimension.

$$S_y = \frac{Sy}{c} \text{ [Eq. (1), Art. 135].}$$

Therefore

Intensity of stress on an element dA distant y from the neutral axis $= \frac{Sy}{c}$.

Normal force acting on this element $= \frac{S}{c}ydA$.

Total normal force acting on the section $BD = \frac{S}{c}\int ydA$.

Or, since $\int ydA = A\bar{y}$ [Eq. (2), Art. 137],

Total normal force acting on the section $BD = \frac{S}{c}A\bar{y}$,

where A = area of section.

$\bar{y}$ = distance of the center of gravity of the section from the neutral axis.

The normal force acting on BD is the only horizontal force acting on the part of the beam under consideration. Hence, since equilibrium exists (since $\Sigma F_x = 0$),

$$\frac{S}{c}A\bar{y} = 0.$$

Now S, the stress in the outer fiber, is not zero; and A, the area of the section, is not zero.

Therefore

$$\bar{y} = 0.$$

That is, the center of gravity of the section lies in the neutral axis. Expressed in another way, *the neutral axis is a gravity axis*, provided the assumptions of Art. 134 are satisfied.

139. Flexure Formula.—The resisting moment M_r acting on the section BD (Fig. 196) is simply the sum of the moments (with respect to the neutral axis mn) of the forces acting on the elementary areas dA of that section. Since the shearing forces acting on the elementary areas of the section lie in the plane of that section and therefore have no lever arms with respect to the neutral axis in that section, these forces take no part in the summation. Hence the normal forces acting on the elementary areas of the section alone are involved in the summation for the resting moment M_r on that section. With reference to Fig. 196,

Normal force acting on an element of area $= \frac{S}{c}ydA$.

Moment of this force with respect to $mn = \frac{S}{c}y\,dAy = \frac{S}{c}y^2dA.$

Σ Moments of the normal forces acting on the section $=$

$$M_r = \frac{S}{c}\int y^2dA.$$

Or, since $\int y^2dA = I$ equals the moment of inertia of the section with respect to the neutral axis mn,

$$M_r = \frac{SI}{c}. \tag{a}$$

In Art. 119, it was shown that $M = M_r$ where M equals the bending moment at that section. Hence Eq. (a) becomes

$$M = \frac{SI}{c}. \tag{9}$$

Equation (9) is called the *flexure formula.**

The flexure formula is an important formula since it gives us a relation between M (the bending moment at a section) and S (the intensity of the stress this bending moment induces in the outer fibers at that section).

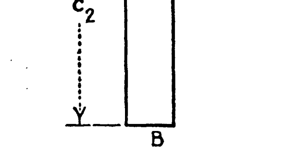


FIG. 197.

140. Outermost Fiber.—Note that the flexure formula [Eq. (9)] contains c, the distance of the outer fiber from the neutral axis. There are, however, two sets of outer fibers, one above and one below the neutral axis. When the neutral axis is *not* an axis of symmetry, the two outer fibers are not equidistant from the neutral axis. Thus (Fig. 197) c_1 does not equal c_2. If the intensity of stress in the outer fibers at D is in question, $c = c_1$; if at B, $c = c_2$. That is, c equals distance from the neutral axis of the outer fiber under consideration.

In this chapter we are concerned with the maximum intensity of the fiber stress at a section. Now the maximum intensity of the fiber stress at a section occurs in the fiber that is most remote from the neutral axis (at B in Fig. 197). Hence, unless a statement to the contrary is made, *the outer fiber should be interpreted to mean the outermost fiber.*

* "Flexure" means "pertaining to bending."

Example.—Referring to Fig. 197, let $I_g = 7.78$ in.[4] and $c_2 = 3.13$ (see Fig. 195). Find the stress in the outermost fiber if $M = 24{,}850$ in.-lb.

$$M = \frac{SI}{c}.$$

Or

$$24{,}850 = S \times \frac{7.78}{3.13}.$$

Therefore

$$S = 10{,}000 \text{ lb./sq. in.}$$

Given $c_1 = 1.37$ in., find the stress in the outer fiber at D.

Ans. $S = 4370$ lb./sq. in.

141. Section Modulus.—The ratio I/c contained in the flexure formula $M = SI/c$, is called the *section modulus* of the beam and

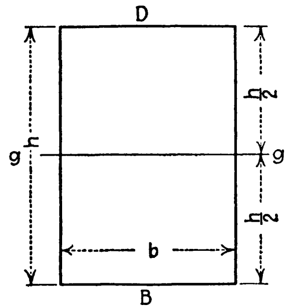


FIG. 198.

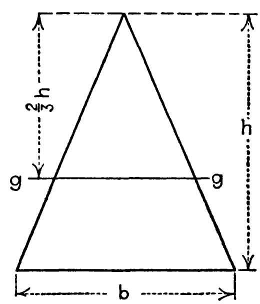


FIG. 199.

will be designated by Z. That is,

$$Z = \frac{I}{c}$$

where I = moment of inertia with respect to the neutral axis (gravity axis).

c = distance of the *outermost* fiber from the neutral axis of the section.*

The section moduli of beams of various shapes and sizes may be computed and tabulated. Such a table will be found useful. For a rectangular or a circular section, the general expression for the section modulus should be remembered.

* In some handbooks, the section modulus is designated by S. Since in this book S is used for fiber stress, the section modulus will be designated by Z.

Rectangular Section. Fig. 198.—With respect to the gravity axis gg

$$I_g = \frac{1}{12}bh^3 \qquad \text{and} \qquad c = \frac{h}{2}.$$

Therefore

$$Z = \frac{I}{c} = \frac{1}{6}bh^2.$$

Triangular Section. Fig. 199:

$$I_g = \frac{1}{36}bh^3, \qquad c = \frac{2}{3}h.$$

Therefore

$$Z = \frac{I}{c} = \frac{1}{24}bh^2.$$

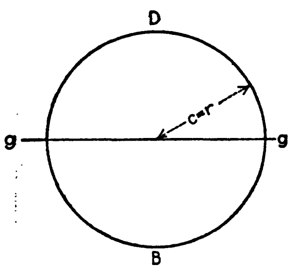


FIG. 200.

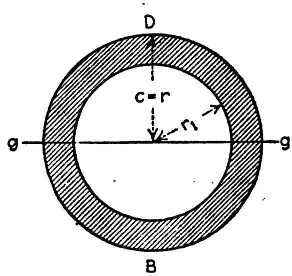


FIG. 201.

Circular Section. Fig. 200:

$$I_g = \frac{1}{4}\pi r^4 \qquad \text{and} \qquad c = r.$$

Therefore

$$Z = \frac{I}{c} = \frac{1}{4}\pi r^3.$$

Section of a Hollow Cylinder. Fig. 201.—Let r equal outer radius, and r_1 equal inner radius.

$$I_g = \frac{\pi}{4}(r^4 - r_1^4) \qquad \text{and} \qquad c = r.$$

Therefore

$$Z = \frac{I}{c} = \frac{\pi}{4}\frac{(r^4 - r_1^4)}{r}.$$

Note that c equals the outer radius and equals the distance of *outermost* fiber from the neutral axis (gravity axis gg).

Note.—The moment of inertia of a plane section is of the fourth dimension since the unit of length (usually the inch) is used four times in a product. Hence the section modulus is of the third dimension. For instance, if $I = 300$ in.^{4} and $c = 6$ in., then

$$Z = \frac{I}{c} = \frac{300}{6} = 50 \text{ in.}^3$$

Example I.—The bending moment at a particular section in a rectangular wooden beam is $M = 51{,}200$ in.-lb. Dimensions of section, 6 in. by 8 in. Find intensity of stress in the outer fiber at this section.

$$M = \frac{SI}{c}.$$

Or

$$51{,}200 = S \times \tfrac{1}{6} \times 6 \times 8^2.$$

Therefore

$$S = 800 \text{ lb./sq. in.}$$

Example II.—Referring to Fig. 174, find the intensity of stress in the outer fibers at D due to the live load if the beam is an I-beam whose section modulus is $I/c = 44.8$ in.^{3} Also at C.

$$M_D = 27.67 \text{ ft.-tons} = 27.67 \times 2000 \times 12 = 664{,}200 \text{ in.-lb.}$$

Therefore

$$S \times 44.8 = 664{,}200.$$

Or

$$S = 14{,}820 \text{ lb./sq. in. (at } D).$$

$$M_C = 17 \text{ ft.-tons} = 408{,}000 \text{ in.-lb.}$$

Therefore

$$S \times 44.8 = 408{,}000.$$

Or

$$S = 9100 \text{ lb./sq. in. (at } C).$$

Problem 119.—If 18,000 lb./sq. in. is the maximum allowable stress in an I-beam, find the maximum bending moment a 20-in. I-beam can carry. Given $I = 1169.5$ in.^{4} *Ans.* $M = 175{,}400$ ft.-lb.

142. Equation of Safe Loading. Prismatic Beam.—If the flexure formula

$$M = \frac{SI}{c} \tag{a}$$

is applied to a given beam under a given loading, it will be found that S, the stress in the outer fiber, will be different for different sections. The section where S is a maximum is called the *dangerous section.* If the beam fails in flexure (bending), it will fail at the dangerous section. Hence it is important to be able to locate the dangerous section in a beam under a given loading

In this chapter, prismatic beams are considered. In a prismatic beam I and c are constants and therefore I/c is a constant. With I/c a constant, it follows from Eq. (*a*) that S is directly proportional to M and is a maximum when M is a maximum. That is, S is a maximum when $M = M_m$. Hence *for a prismatic beam, the dangerous section is the section of maximum bending moment.*

Assume now that for a prismatic beam there is one unknown quantity. This unknown quantity may be a moment, a load, a span, the section modulus, or any one dimension of the beam. If this unknown quantity is to be determined so that S, the stress in the outer fiber at the dangerous section, is to reach its maximum allowable value S', then the equation

$$\frac{S'I}{c} = M_m \tag{10}$$

must be satisfied. Equation (10), therefore, may be called *the equation of safe loading* for a prismatic beam.

Note.—Equation (10) frequently will be written $SI/c = M_m$, where S equals the allowable stress in the outer fiber. In this book, when ambiguity is likely to result, S' will be used to denote the allowable stress in the outer fiber.

Example I. Safe Load at the Middle of a Simple Beam on End Supports. Fig. 202.—Construct the shear and moment diagrams in the usual way. The maximum moment occurs at C and is

$$M_m = \frac{P}{2}\cdot\frac{L}{2} = \frac{PL}{4}.$$

Therefore the equation of safe loading is [Eq. (10)]

$$\frac{S'I}{c} = \frac{PL}{4}. \tag{a}$$

Illustration.—A wooden beam on end supports is to carry a central load of $P = 2000$ lb. The beam is 12 ft. long and 4 in. wide. The safe fiber stress is given as $S' = 1200$ lb./sq. in.

Required to find the minimum safe height of the beam.

$$\frac{I}{c} = \frac{1}{6}bh^2 = \frac{1}{6} \times 4 \times h^2 = \frac{2}{3}h^2 \text{ (see Art. 141).}$$

$$M_m = \frac{PL}{4} = \frac{2000 \times 12}{4} = 6000 \text{ ft.-lb.} = 72{,}000 \text{ in.-lb.}$$

Therefore [Eq. (10)],

$$1200 \times \tfrac{2}{3}h^2 = 72{,}000 \qquad \text{or} \qquad h = 9.49 \text{ in.}$$

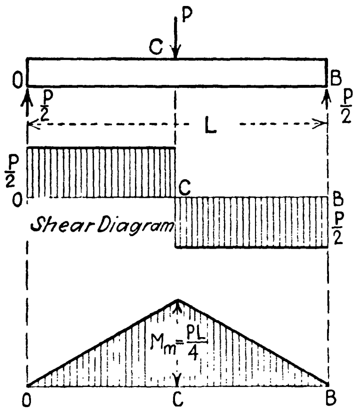


Fig. 202.

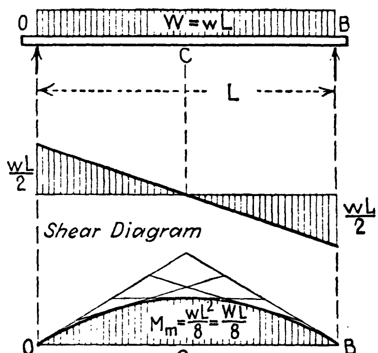


Fig. 203.

Example II. Safe Load Uniformly Distributed over a Simple Beam on End Supports. Fig. 203.—Construct the shear and moment diagrams. The maximum moment occurs at C and is

$$M_m = \frac{wL}{2} \cdot \frac{L}{2} - \frac{wL}{2} \times \frac{L}{4} = \frac{wL^2}{8} = \frac{WL}{8}.$$

Therefore equation of safe loading is [Eq. (10)]

$$\frac{S'I}{c} = \frac{wL^2}{8} = \frac{WL}{8}. \qquad (b)$$

Illustration.—A round bar of steel, 2 in. in diameter, rests *horizontally* on end supports. Required the maximum length L this bar may have if, due to its own weight, the fiber stress is not to exceed 20,000 lb./sq. in. Steel weighs 490 lb./cu. ft.

$$w = \text{weight per running foot} = \frac{\pi \times 1^2}{144} \times 1 \times 490 = 10.68 \text{ lb.}$$

per foot of length.

$$M_m = \frac{1}{8}wL^2 = \frac{10.68L^2}{8} = 1.335L^2 \text{ ft.-lb. } (L \text{ in feet}).$$
$$= 16.02L^2 \text{ in.-lb.}$$
$$\frac{I}{c} = \frac{1}{4}\pi r^3 = 0.786 \text{ in.}^3 \text{ (Art. 141).}$$

Therefore [Eq. (10)]

$$20{,}000 \times 0.786 = 16.02L^2 \qquad \text{or} \qquad L = 31.3 \text{ ft.}$$

Example III. Safe Load on End of Cantilever. Fig. 204.—Construct shear and moment diagrams. Maximum moment occurs at B and is

$$M_m = PL.$$

Therefore equation of safe loading is

$$\frac{S'I}{c} = PL. \tag{c}$$

Illustration.—A hollow iron pipe, one end of which is built into a wall, extends 6 ft. out from the wall. Outer radius of

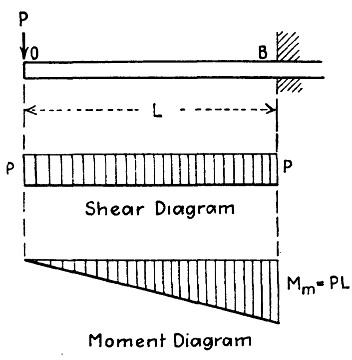


FIG. 204.

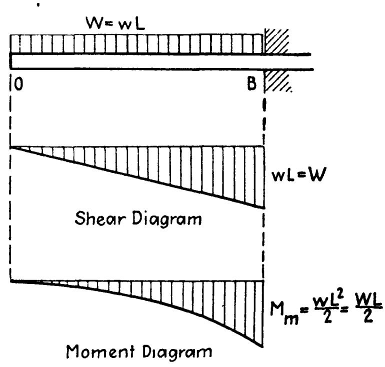


FIG. 205.

pipe is $r = 2$ in., and inner radius is $r_1 = 1.75$ in. With the weight of the pipe neglected, what weight may be hung from the free end if the fiber stress is not to exceed $S' = 9000$ lb./sq. in.?

$$M_m = PL = P \times 6 \text{ ft.-lb.} = 72P \text{ in.-lb.}$$
$$\frac{I}{c} = \frac{1}{4}\pi\frac{(r^4 - r_1^4)}{r} = \frac{1}{4}\pi\frac{(2^4 - 1.75^4)}{2} = 2.605.*$$

* When a slide rule is used to evaluate an expression of the form $a^2 - b^2$, or of the form $a^4 - b^4$, proceed as follows:

$$a^2 - b^2 = (a + b)(a - b) = \text{product of two numbers.}$$
$$a^4 - b^4 = (a^2 + b^2)(a + b)(a - b) = \text{product of three numbers.}$$

Therefore [Eq. (10)]

$$9000 \times 2.605 = 72P \qquad \text{or} \qquad P = 326 \text{ lb.}$$

Example IV. Safe Load Uniformly Distributed over Cantilever. Fig. 205.

$$M_m = \frac{WL}{2} = \frac{wL^2}{2}.$$

Therefore equation of safe loading is

$$\frac{S'I}{c} = \frac{WL}{2} = \frac{wL^2}{2}. \tag{d}$$

Illustration.—An iron bar, one end of which is built into a wall, extends 6 ft. out from the wall. The bar has a triangular section ($b = 3$ in., $h = 4$ in.). What uniformly distributed load can this cantilever safely carry if $S' = 18{,}000$ lb./sq. in.?

$$M_m = \frac{WL}{2} = \frac{W \times 6}{2} \text{ ft.-lb.} = 36W \text{ in.-lb.}$$

$$\frac{I}{c} = \frac{1}{24}bh^2 = \frac{1}{24} \times 3 \times 4^2 = 2 \text{ in.}^3$$

Therefore [Eq. (10)]

$$18{,}000 \times 2 = 36W \qquad \text{or} \qquad W = 1000 \text{ lb.}$$

Steel weighs 490 lb./cu. ft. The weight of the bar equals

$$\frac{1}{2} \times \frac{3 \times 4}{144} \times 6 \times 490 = 122.5 \text{ lb.}$$

Hence, in addition to its own weight, the bar can carry

$$1000 - 122.5,$$

or 877.5, lb. uniformly distributed.

PROBLEMS

120. A rectangular wooden beam on end supports is to carry a central load of $P = 2400$ lb. The beam is to be 12 in. high and 20 ft. long. $S = 1000$ lb./sq. in. Required to find the width of the beam. *Ans.* $b = 6$ in.

121. In addition to the load given in Problem 120, the beam is to carry a load of $W = 4800$ lb. uniformly distributed over the whole beam. Required to find the width. *Ans.* $b = 12$ in.

143. Résumé of the Four Simple Cases.—The four simple cases considered in Art. 142 arise frequently in one form or another. The following table should be carefully studied.

To complete the table the maximum deflection of each beam is given in the last column. Deflection will be considered later.

Beam		Maximum moment (1)	Equation of safe loading (2)	Safe load (3)	Relative strength (4)	Maximum deflection (5)
Cantilever	P at end (Fig. 204)	PL	$S'\frac{I}{c} = PL$	$P = \frac{S'}{L}\frac{I}{c}$	1	$\frac{1}{3}\frac{PL^3}{EI}$
	W uniformly distributed (Fig. 205)	$\frac{WL}{2}$	$S'\frac{I}{c} = \frac{WL}{2}$	$W = 2\frac{S'}{L}\frac{I}{c}$	2	$\frac{1}{8}\frac{WL^3}{EI}$
Beam on end supports	P at middle (Fig. 202)	$\frac{PL}{4}$	$S'\frac{I}{c} = \frac{PL}{4}$	$P = 4\frac{S'}{L}\frac{I}{c}$	4	$\frac{1}{48}\frac{PL^3}{EI}$
	W uniformly distributed (Fig. 203)	$\frac{WL}{8}$	$S'\frac{I}{c} = \frac{WL}{8}$	$W = 8\frac{S'}{L}\frac{I}{c}$	8	$\frac{5}{384}\frac{WL^3}{EI}$

In the fourth column, the relative strength of the beam is given. For instance, if a beam is used as a cantilever and the safe end load is $P = 2$ tons, then this same beam if resting on end supports can safely carry a uniformly distributed load of $W = 8 \times 2 = 16$ tons.

Note.—Column 3 gives the safe load the beam can carry on the assumption that flexure governs. Shear will be considered later.

144. Moment Diagram as a Stress Diagram.—For a prismatic beam, S, the stress in the outer fiber at a section, is proportional to M, the bending moment at that section (Art. 142). Accordingly, the moment diagram for a loaded prismatic beam serves also as a fiber stress diagram. This interpretation makes the moment diagram useful, since the moment diagram considered as a stress diagram shows how the stress in the outer fiber of a prismatic beam varies from section to section. For instance (Fig. 204), the moment diagram shows that the fiber stress in a prismatic cantilever carrying an end load increases uniformly from zero at O to a maximum at B.

145. Economical Section.—Let it be required to select a prismatic beam to carry a given system of loads. Assume for the present that flexure governs. For the beam to be safe in flexure, the section modulus must not be less than that determined from the equation of safe loading $S'I/c = M_m$. That is, for safety,

$$\text{Min. } \frac{I}{c} = \frac{M_m}{S'}$$

where M_m = maximum bending moment in the beam.

S' = safe unit stress in the outermost fiber.

Effect of Shape of Section.—The section modulus of a beam of a given sectional area depends upon the shape of the section. Figure 206 gives three sections of the same area. Note the difference in the values of the section modulus. In general, in so far as the strength of a beam depends upon the section modulus, *economy of material will result if the section is so shaped that the greater part of the area is as far from the neutral axis as practicable.*

Note.—Strength in flexure is not the only requirement for a beam. The beam must have the necessary strength in shear

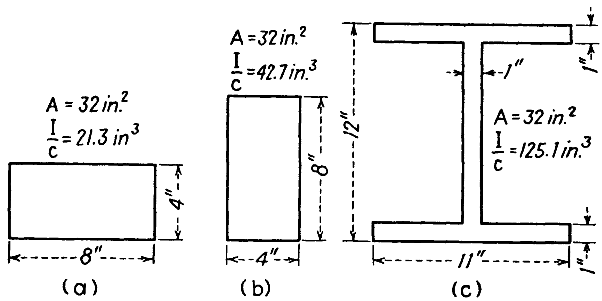


FIG. 206.

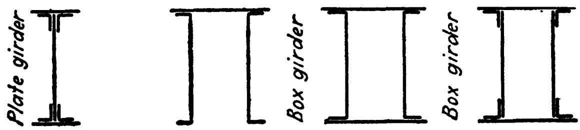
(shear will be considered later) and frequently must also have a certain degree of rigidity. In particular cases, special requirements must be satisfied. If the section of a beam is so shaped that the beam meets all requirements as to strength, rigidity, etc., and at the same time contains as little material as practicable, the section is called an *economical section.* This assumes that the cost of the beam is proportional to the amount of material in that beam.

Wooden beams are sawed to shape and are rectangular for the simple reason that it does not pay to shape them in any other way. Steel beams, however, are rolled to shape. By varying the shapes and the sizes of the rolls, many rolled sections become available (see Appendix).

In practice, if a steel beam is to be used, I-beams usually are preferred. Or, if an I-beam is not suitable, a beam is "built up" of various shapes in such a way that the beam has the required section modulus and at the same time contains as little material

as is practicable. Figure 207 illustrates four built-up sections commonly used.

Stock or Commercial Sizes.—A beam made to order will cost more, as a rule, than one that can be ordered from stock, even though the stock size contains a little more material than theoretically required. Hence, when a beam is selected, it will be economical to use the nearest stock size available.



Built-up Sections
FIG. 207.

146. Wooden Beams, Commercial Sizes.—The following table gives the commercial sizes adopted by the Southern Pine Association.

NOMINAL DIMENSIONS*

Height, in.	Width, in.
4	2, 4
6, 8, 10, 12, 14, 16, 18	2, $2\frac{1}{2}$, 3, 4, 6, 8, 10, 12, 14, 16, 18

* The actual dimensions of dressed beams are slightly smaller. For instance, an 8- by 12-in. beam when dressed is $7\frac{1}{2}$ by $11\frac{1}{2}$ in.

Accordingly, a 4- by 7-in. beam is not a commercial size. Such a beam will cost more, as a rule, than a 4- by 8-in. beam.

Example.—A rectangular wooden beam on end supports is 12 ft. long and is to carry a uniformly distributed load of

$$W = 4 \text{ tons.}$$

The safe fiber stress is $S' = 1200$ lb./sq. in. Determine the beam if $h = 3b$. Then select a commercial size.

The equation of safe loading is, since $M_m = WL/8$,

$$\frac{WL}{8} = S'\frac{I}{c}$$

or

$$\frac{(4 \times 2000) \times (12 \times 12)}{8} = 1200\frac{I}{c}.$$

Therefore

$$\frac{I}{c} = 120 \text{ in.}^3$$

For a rectangular beam,

$$\frac{I}{c} = \frac{1}{6}bh^2 \text{ (Art. 141).} \qquad \text{Or, since} \qquad h = 3b,$$

$I/c = \frac{3}{2}b^3$. Therefore $120 = \frac{3}{2}b^3$. Or $b = 4.3$ in. and

$$h = 12.9 \text{ in.}$$

The nearest commercial size is either a 6- by 12-in. or a 4- by 14-in. beam.

For a 6- by 12-in. beam,

$$\frac{I}{c} = \frac{1}{6} \times 6 \times 12^2 = 144 > 120.$$
$$A = 72 \text{ sq. in.}$$

For a 4- by 14-in. beam,

$$\frac{I}{c} = \frac{1}{6} \times 4 \times 14^2 = 131 > 120.$$
$$A = 56 \text{ sq. in.}$$

Either beam has more than the required strength in flexure. The second beam, however, has the smaller sectional area A. It will be seen later that the second beam must be investigated for strength in shear. A 4- by 14-in. beam may cost more (on account of its height) than a 6- by 12-in. beam. If that is the case, a 6- by 12-in. beam is the economical beam.

147. I-beams.—There are two types of I-beams rolled at present—the *standard I-beam* and the *Wide Flange* (WF) *I-beam* Figure 208*a* shows a standard I-beam, and Fig. 208*b* a WF I-beam. The two beams shown are of the same height and weigh approximately the same. Note that in the standard beam the flange tapers. In the WF beam, this is not the case. Note also that the WF beam has a greater section modulus than the standard beam. As a rule, the WF beam is stronger in flexure (but weaker in shear) than a standard beam of the same height and weight.

In the Appendix two tables are given, one for the standard and one for the WF I-beam. These tables give the properties of the various I-beams—dimensions of beam, weight per running foot, moment of inertia, section modulus, etc. Tables are also

given for standard channels and angles. If then a property of one of the rolled sections is required, the student is expected to consult the tables given in the Appendix, or a Steel Handbook.

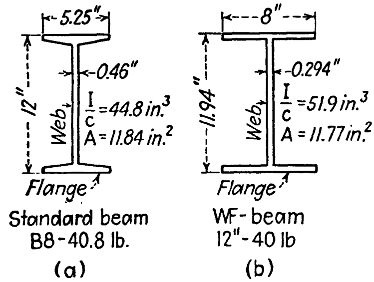


FIG. 208.

Illustration.—A steel beam requires a section modulus of $Z = 30$ in.³ Select a standard I-beam.

Consult the table for standard I-beams. In the column under Z (axis 1–1), the nearest section modulus above that required (above 30) is found to be 31.6. Hence a 10-in. 40-lb. beam is satisfactory. Note, however, that a 12-in. 31.8-lb. beam has a section modulus of 36. Since the 12-in. beam is lighter than the 10-in. beam, the 12-in. beam is the more economical.

Note.—When no special requirements are imposed, the lightest beam whose section modulus is above that required should be selected as a rule.

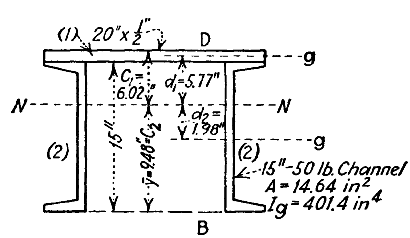


FIG. 209.

148. Built-up Section.—When a rolled beam is used, the properties of its section are obtained from tables. When a section is "built up," the properties of the section must be calculated, the properties of the constituent parts (channels, I-beams, angles, etc.) being obtained from tables. This now will be illustrated.

Example. Fig. 209.—(1) The area of the section is

Cover plate:

$$20 \times \tfrac{1}{2} \qquad = 10$$

Channels:

$$2 \times 14.64 = 29.28$$

Therefore $\qquad A = \overline{39.28}$ sq. in.

(2) Position of neutral axis. Take B as an axis and use the formula [Eq. (7), Art. 137].

$$(\Sigma A)\bar{y} = \Sigma(Ay).$$

Or

$$39.28\bar{y} = 10 \times 15.25 + (2 \times 14.64) \times 7.5.$$

Or

$$\bar{y} = 9.48 \text{ in.}$$

This locates the neutral axis.

Therefore

$$c_1 = 15.5 - 9.48 = 6.02 \text{ in.}$$

$$c_2 = 9.48 \text{ in.} = c.$$

(3) Moment of inertia with respect to the gravity axis N [Eq. (8), Art. 137].

Cover plate:

$$\tfrac{1}{12} \times 20 \times (\tfrac{1}{2})^3 + (20 \times \tfrac{1}{2}) \times \overline{5.77^2} = \quad 333.1$$

Channels:

$$2(401.4 + 14.64 \times \overline{1.98^2}) = \quad 917.6$$

Therefore $\qquad I_N = 1250.7 \text{ in.}^4$

(4) Section modulus:

$$Z = \frac{I}{c} = \frac{1250.7}{9.48} = 132 \text{ in.}^3$$

Note.—If 18,000 lb./sq. in. is taken as the safe fiber stress, then the maximum moment this beam can safely carry is

$$M_m = S\frac{I}{c} = 18{,}000 \times 132 = 2{,}375{,}000 \text{ in.-lb.}$$

DESIGN OF BEAMS FOR FLEXURAL STRENGTH

149. Dead Load. Live Load.—The weight of the beam is a load that frequently must be considered when selecting a beam.

For convenience, the weight of the beam will be called the *dead load,* and the load or system of loads placed on the beam (all loads except the dead load) will be called the *live load.*

In some cases, the weight of the beam is so small when compared with the live load that it may be neglected. Moreover, it is seldom that a beam of standard or commercial size can be found whose strength is exactly that required to carry the live load. That is, it will be necessary, as a rule, to select a beam whose strength is slightly greater than that required to carry the live load. This excess strength often is more than sufficient to take care of the beam's own weight. An experienced designer usually knows when the weight of the beam should be given special consideration.

150. Method of Procedure. Weight of Beam Neglected.—In Art. 130, it was shown how to find the maximum moment in a beam due to the live load. Assuming that this moment (now designated by M_1) has been found, we proceed as follows:*

(5) $SI/c = M_1$. Or

$\frac{I}{c} = \frac{M_1}{S} = Z_1 =$ section modulus required to carry the live load.

(6) Select a beam whose section modulus is equal to or preferably slightly larger than that required to carry the live load. That is, if Z equals section modulus of beam selected, then Z should be equal to or preferably slightly larger than Z_1.

Note 1.—If a wooden beam is to be selected, consult the table of Art. 146 for commercial sizes. If an I-beam is to be selected, consult the tables given in the Appendix.

Note 2.—The unit stress usually is designated. If not,
For structural steel use:

$$S = 18{,}000 \text{ lb./sq. in.}$$

For wood use:

$$S = 1200 \text{ lb./sq. in.}$$

Example I.—A standard I-beam is to be used to carry the load given in Example I of Art. 130. The maximum bending moment due to the live load was there found to be

$$M_1 = 27.69 \text{ ft.-tons} = 27.69 \times 2000 \times 12 = 664{,}000 \text{ in.-lb.}$$

Neglect the weight of the beam. Take $S = 18{,}000$ lb./sq. in.

* The complete analysis of a beam includes the analysis given in Art. 130.

(5) $SI/c = 664{,}000$. Or

$$\frac{I}{c} = \frac{664{,}000}{18{,}000} = 36.9 \text{ in.}^3 = Z_1$$

where Z_1 = section modulus required to carry the live load.

(6) By consulting the table (see Appendix), it will be found that the first standard beam whose section modulus is greater than 36.9 is a 12-in. 35-lb. beam whose $I/c = 37.8$. That is,

$$Z = 37.8 = \text{section modulus of beam selected.}$$

Note.—The beam selected has a section modulus 0.9 larger than that required to carry the live load. An experienced designer would know that this is sufficient to take care of the weight of the beam in this particular case. Therefore take a 12-in. 35-lb. standard I-beam.

Example II.—A rectangular wooden beam on end supports is 8 ft. long and is to carry a central concentrated load of

$$P = 2500 \text{ lb.}$$

Make the width of the beam not less than half its height. Take $S = 1200$ lb./sq. in. Neglect the weight of the beam.

Maximum moment is

$$M_1 = \tfrac{1}{4}PL = \tfrac{1}{4} \times 2500 \times 8 \times 12 = 60{,}000 \text{ in.-lb.}$$

(5) $SI/c = 60{,}000$.

$$\frac{I}{c} = \frac{60{,}000}{1200} = 50 = Z_1.$$

Putting $Z_1 = \frac{1}{6}bh^2$ (Art. 141) and $b = h/2$,

$$50 = \tfrac{1}{12}h^3.$$

Therefore

$$h = 8.43 \text{ in.} \qquad \text{and} \qquad b = 4.22 \text{ in.}$$

Since the width of the beam is to be not less than half its height, the nearest commercial size is a 6- by 8-in. beam.

(6) Try a 6- by 8-in. beam.

$$Z = \tfrac{1}{6}bh^2 = \tfrac{1}{6} \times 6 \times 8^2 = 64 > 50.$$

That is, Z is greater than Z_1. Therefore take a 6- by 8-in. beam.

Problem 122.—A standard I-beam on end supports is to be used to carry a load of $P = 20$ tons. The beam is to be 16 ft. long and the load is to be placed 4 ft. from the left end. $S = 18{,}000$ lb./sq. in. Select a suitable beam of minimum height. *Ans.* 15-in. 60.8-lb. beam.

151. Weight of Beam Considered.—If the weight of the beam is to be considered, we may proceed as follows:

Neglect the weight of the beam and determine Z_1, the section modulus required to carry the live load (Art. 150).

As a first trial select a beam whose section modulus Z is a little larger than Z_1. An experienced designer usually knows *about* how much allowance must be made in a particular case to take care of the beam's own weight.

Consider the weight of the trial beam as a uniformly distributed load superimposed upon the live load and determine Z', the section modulus required to carry both loads (dead and live load).

If Z' (section modulus required to carry both the dead and the live load) is equal to or *slightly less* than Z (section modulus of trial beam), the beam is satisfactory. If not, make another trial.

Illustration.—A standard I-beam, 30 ft. long, is to rest on end supports and is to carry a load of 64,000 lb. uniformly distributed over the beam. It is required to select the beam of minimum height. Its own weight is to be considered. Given

$$S = 18{,}000 \text{ lb./sq. in.}$$

With the weight of the beam neglected, the maximum moment is

$$M_1 = \tfrac{1}{8}WL = \tfrac{1}{8} \times 64{,}000 \times (30 \times 12) = 2{,}880{,}000 \text{ in.-lb.}$$

and the section modulus required to carry this moment is

$$Z_1 = \frac{M_1}{S} = \frac{2{,}880{,}000}{18{,}000} = 160 \text{ in.}^3$$

As a first trial, select a 20-in. 100-lb. beam with section modulus

$$Z = 164.8 \text{ in.}^3$$

This beam weighs 100 × 30, or 3000, lb. Hence the total load this beam must carry is 67,000 lb. (uniformly distributed).

The maximum bending moment due to both loads is

$$M = \tfrac{1}{8} \times 67{,}000 \times 30 \times 12 = 3{,}015{,}000 \text{ in.-lb.}$$

and the section modulus required to carry both loads is

$$Z' = \frac{3{,}015{,}000}{18{,}000} = 167.4 \text{ in.}^3$$

Now Z' is greater than Z; *i.e.*, 167.4 is greater than 164.8. Hence, the section modulus required to carry both loads is greater than that of the beam selected. It is necessary, therefore, to select a

stronger beam. The next stronger beam is a 24-in. 79.9-lb. beam whose section modulus is 173.9 in.^{3} Since this beam weighs less than the first beam, it is not necessary to test it for its own weight.

152. Weight of Beam. Simplified Method of Procedure.—Usually the weight of the beam is small when compared with the live load the beam is to carry. In such a case, when the weight of the beam is considered, the method of procedure often may be greatly simplified.

Note 1.—When the weight of the beam is superimposed upon the live load, the position of the section of maximum moment due to both loads (live and dead load) may *not* be the same as that due to the live load alone. For instance, Fig. 210*a* shows a beam carrying a uniformly distributed load W over the left half of its length. Figure 210*b* gives the shear diagram for the load W alone, and Fig. 210*c* gives the shear diagram for the weight of the beam alone. Note that, when the two shear

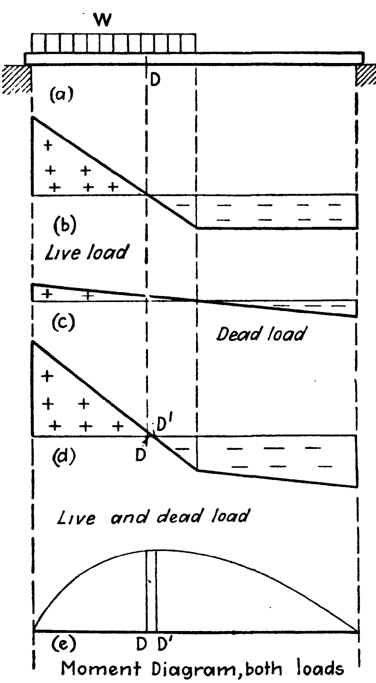


FIG. 210.

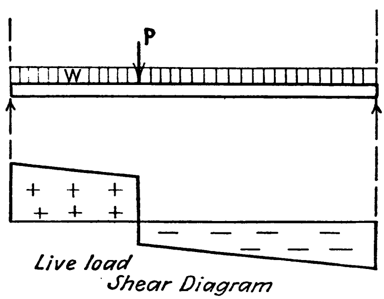


FIG. 211.

diagrams are combined (Fig. 210*d*), the shear diagram for the combined loads crosses the base line a little to the right of D and hence the dangerous section for the combined loads is at D'. However, in all ordinary cases (weight of beam small when compared with the live load) no appreciable error will be made if it is assumed that D is the dangerous section due to the combined loads. It should be noted (Fig. 210*e*) that the ordinates to the

moment curve for a distributed load do not vary appreciably in the neighborhood of the maximum ordinate. That is, if the maximum ordinate due to both live and dead load is at D', then this ordinate does not differ appreciably from the ordinate at D (a neighboring point).

If the shear diagram due to the live load alone crosses the base line under a concentrated load (Fig. 211), then, as a rule, the shear diagram due to both loads (live and dead load) also crosses the base line under the concentrated load.

In general, with *weight of beam small when compared with the live load* (the usual case), *the position of the section of maximum moment due to both live and dead load may be taken as that due to the live load alone.*

Note 2.—If M_1 equals the bending moment at a given section due to the live load, and M_2 equals the bending moment at that section due to the dead load, then $M = M_1 + M_2$ equals the bending moment at that section due to both loads. The section modulus the beam must have at that section is

$$Z' = \frac{M}{S} = \frac{M_1 + M_2}{S} = \frac{M_1}{S} + \frac{M_2}{S} = Z_1 + Z_2 \qquad (a)$$

where Z_1 = section modulus required at that section to carry the live load;

Z_2 = that required at that section to carry the dead load.

With the results of Note 1 kept in mind, this suggests the following method of procedure:

Find Z_1, the section modulus required to carry the live load. At the section of maximum moment due to the live load, find Z_2, the section modulus required to carry the dead load. The section modulus required to carry both loads is

$$Z' = Z_1 + Z_2.$$

Note 3.—*In a simple beam on end supports* the maximum bending moment due to a load uniformly distributed over its whole length occurs at the middle of the beam and is

$$M_m = \tfrac{1}{8}wL^2.$$

where w = load per unit length of beam.

If w equals the weight of beam per unit length, this equation gives the maximum bending moment developed in the beam due to its own weight.

Referring to Fig. 212, note that the moment diagram for the weight of the beam (dotted curve) is very flat (when compared with the moment diagram for the live load) and that the ordinates to this diagram do not vary appreciably for the middle half of the beam. This assumes that the weight of the beam is small when compared with the live load (the usual case).

Accordingly, *when determining* Z_2 (the section modulus required for the beam to carry its own weight) *we may take* $M_2 = \frac{1}{8}wL^2$, where w equals the weight of beam per unit length. That is,

$$Z_2 = \frac{\frac{1}{8}wL^2}{S}. \qquad (b)$$

It should be remembered that *Eq.* (*b*) *applies only to simple beams on end supports.* This equation, therefore should *not* be used for overhanging beams or for cantilevers.

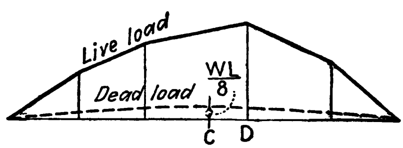


Fig. 212.

Note 4.—For a beam of given length, supported and loaded in a given way, let w equal the weight per unit length of the beam selected as a first trial, and let Z_2 equal the section modulus required for this beam to carry its own weight.

Assume now that for some reason a beam weighing w' per unit length is to be considered. The corresponding section modulus Z_2' may be found from the relation

$$\frac{Z_2'}{Z_2} = \frac{w'}{w} \qquad \text{or} \qquad Z_2' = Z_2 \frac{w'}{w}. \qquad (c)$$

Equation (*c*) is almost self-evident. This equation can easily be proved for a simple beam on end supports. For such a beam,

$$Z_2 = \frac{1}{8}\frac{wL^2}{S} \qquad \text{and} \qquad Z_2' = \frac{1}{8}\frac{w'L^2}{S}.$$

Dividing the second equation by the first, member by member,

$$\frac{Z_2'}{Z_2} = \frac{w'}{w}.$$

Illustration.—If a 40-lb. beam requires a section modulus of $Z_2 = 1.20$ in.³ to carry its own weight, then, under similar conditions, a 45-lb. beam will require a section modulus of

$$Z_2' = Z_2 \frac{w'}{w} = 1.20 \times \frac{45}{40} = 1.35 \text{ in.}^3$$

Note 5.—In the problems that follow, the student should note the effect of the weight of the beam upon the required section modulus. For short beams carrying a fairly heavy live load, it will be found that *the weight of the beam may be neglected.*

153. Résumé.—The design of a beam is one of the important problems in Mechanics of Materials. The method of procedure, therefore, should be carefully noted.

For convenience of reference, the symbols used will be restated.

S = allowable stress in outer fiber.

M_1 = maximum bending moment due to live load alone.

Z_1 = section modulus required to carry live load $= \dfrac{M_1}{S}$.

Z = section modulus of trial beam.

M_2 = bending moment due to weight of beam (at the section of maximum moment due to the live load).

Z_2 = section modulus required for the beam to carry its own weight $= \dfrac{M_2}{S}$.

$Z' = Z_1 + Z_2$ = section modulus required to carry both loads (live and dead load).

Example I.—Consider the weight of the I-beam selected in Example I of Art. 150.

(5) The section modulus required for the beam to carry the live load was there found to be

$$Z_1 = 36.9.$$

(6) A 12-in. 35-lb. beam was selected whose section modulus is

$$Z = 37.8.$$

(7) The beam is a simple beam on end supports and is 15 ft. long (see Fig. 174). Hence (Note 3, Art. 152) the additional moment (due to its own weight) may be taken as

$$M_2 = \tfrac{1}{8}wL^2 = \tfrac{1}{8} \times 35 \times 15^2 = 985 \text{ ft.-lb.} = 11{,}830 \text{ in.-lb.}$$

(8) The section modulus required for the beam to carry its own weight is

$$Z_2 = \frac{11{,}830}{18{,}000} = 0.66.$$

(9) Section modulus required for the beam to carry both the live load and the dead load is

$$Z' = Z_1 + Z_2 = 36.9 + 0.66 = 37.6.$$

Since Z' is less than Z, *i.e.*, since 37.6 is less than 37.8, the beam is satisfactory.

Example II.—Consider the wooden rectangular beam selected in Example II of Art. 150.

(5) $Z_1 = 50.$

(6) A 6- by 8-in. beam was selected whose section modulus is

$$Z = 64.$$

(7) If wood weighs 40 lb./cu. ft., the weight of the beam is ($L = 8$ ft.)

$$W = \frac{6 \times 8}{144} \times 8 \times 40 = 107 \text{ lb.}$$

Therefore

$$M_2 = \frac{1}{8}WL = \frac{1}{8} \times 107 \times 96$$
$$= 1284 \text{ in.-lb.}$$

(8) $Z_2 = \dfrac{1284}{1200} = 1.07.$

(9) $Z' = 50 + 1.07 = 51.07 < 64.$

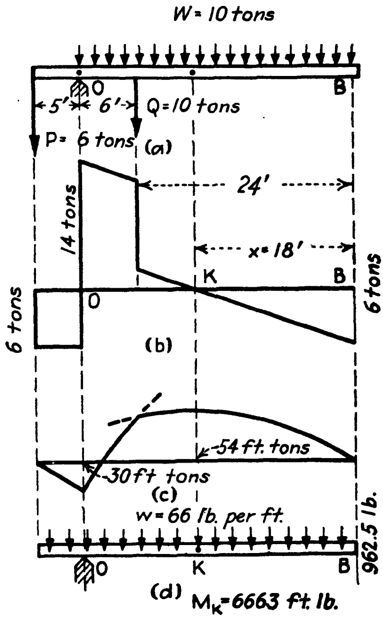


FIG. 213.

That is, Z' is less than Z. Therefore beam is satisfactory.

Example III.—Select a WF beam to carry the loading shown in Fig. 213. Take 18,000 lb./sq. in. as the safe fiber stress. Verify all results given. Neglecting the weight of the beam,

(1) $R_O = 20$ tons, $R_B = 6$ tons.

(2) Shear diagram (Fig. 213*b*):

$$V_m = 14 \text{ tons.}$$

(3) Maximum moment:

$$M_O = -30 \text{ ft.-tons.}$$
$$x = 18 \text{ ft.} \qquad M_1 = 54 \text{ ft.-tons}, = M_K.$$

(4) Moment diagram (Fig. 213*c*).

Note.—It was shown in Art. 144 that the moment diagram serves also as a stress diagram. Hence Fig. 213*c* indicates how the fiber stress due to the live load varies from section to section.

(5) $Z_1 = \dfrac{54 \times 2000 \times 12}{18{,}000} = 72.$

(6) Try a WF 16-in. 45-lb. beam. $Z = 72.4.$

Consider now the weight of the beam (Fig. 213*d*).

(7) The weight of the beam is a load uniformly distributed. To find M_2, the moment at K, first find R_B, the reaction at B due to the weight of the beam.

$\Sigma M_O = 0$ gives*

$$R_B \times 30 - (45 \times 35) \times 12.5 = 0;$$
$$R_B = 656 \text{ lb.} = \text{right reaction.}$$

Therefore

$$M_2 = 656 \times 18 - \frac{45 \times 18^2}{2} = 4518 \text{ ft.-lb.}$$

(8) $Z_2 = \dfrac{M_2}{S} = \dfrac{4518 \times 12}{18{,}000} = 3.01.$

(9) $Z' = Z_1 + Z_2 = 72 + 3.01 = 75.01.$

Since Z' is greater than Z $(75.01 > 73.7)$, beam is unsatisfactory. A stronger beam is required.

(6) Try an 18-in. 47-lb. WF beam. $Z = 82.3.$

(8) The section modulus required for this beam to carry its own weight may be found from the relation (see Note 4, Art. 152)

$$Z_2' = Z_2 \frac{w'}{w}.$$

Or

$$Z_2' = 3.01 \times \frac{47}{45} = 3.14.$$

(9) $Z' = Z_1 + Z_2' = 72 + 3.14 = 75.14,$

which is less than 82.3. Therefore take a 47-lb. WF beam.

Example IV.—Select a WF beam to carry the load given in Fig. 214. $S = 18{,}000$ lb./sq. in. Verify all results given.

(1) $R_O = 5.85$ tons. $R_B = 5.15$ tons.

(2) Shear diagram (Fig. 214*b*):

$$V_m = 5.18 \text{ tons.}$$

* The resultant of the weight of the beam acts at the middle of the beam or 17.5 ft. from the end or 12.5 ft. from O.

(3) Maximum moment:

$$M_O = -0.667 \text{ ft.-ton.}$$
$$x = 7.45'. \qquad M_1 = 22.56 \text{ ft.-tons}, = M_k$$

(4) Moment diagram (Fig. 214c).
(5) $Z_1 = 30.1$ in.^{3}
(6) Try a 12-in. 25-lb. WF beam. $Z = 30.9$.

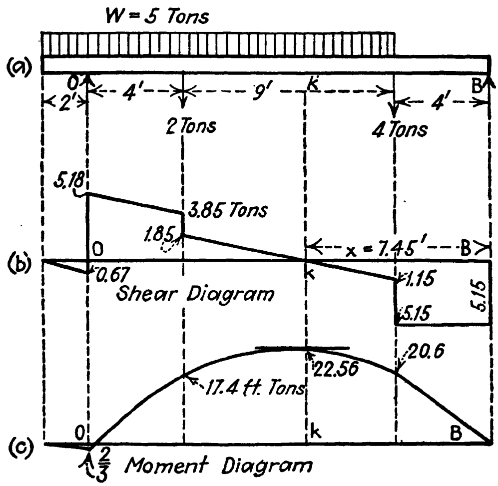


FIG. 214.

(7) Weight of beam. $\Sigma M_O = 0$ gives*

$$R_B \times 17 - (25 \times 19) \times 7.5 = 0. \qquad R_B = 209.5 \text{ lb.}$$

$$M_2 = M_k = 209.5 \times 7.45 - \frac{25 \times \overline{7.45}^2}{2} = 867 \text{ ft.-lb.}$$

(8) $Z_2 = \dfrac{867 \times 12}{18{,}000} = 0.578.$

(9) $Z' = Z_1 + Z_2 = 30.68.$

Beam is satisfactory. Why?

PROBLEMS

123. Design an American Standard steel I-beam for the loading and supports shown in Fig. 215. Use 9 tons/sq. in. as the allowable stress in bending. *Ans.* 15-in. 45 lb. beam.

* The resultant of the weight of the beam acts at the middle of the beam, *i e.*, 9.5 ft. from the left end, or 7.5 ft. from O. Weight of beam = 25 × 19 lb.

124. In the preceding problem, recompute for the effect of the weight of the beam. *Ans.* The additional section modulus required is 1.26 in.^{3}

125. A wooden joist is 3 in. wide and the allowable bending stress for the material is 1200 lb./sq. in. The loading and supports are shown in Fig. 216. Find the "commercial" size of the beam. *Ans.* h = 12 in.

126. Fig. 217. Compute the spacing of joists 2 in. wide, 6 in. deep, and 16 ft. long to carry a dwelling-house floor load of 45 lb./sq. ft. of floor area.

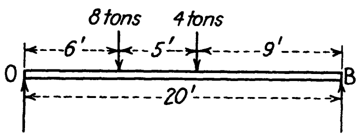


FIG. 215.

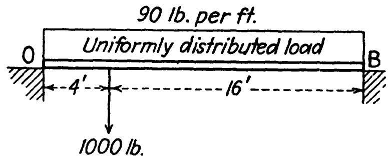


FIG. 216.

Use 1600 lb. per sq. in. as the safe value of the bending stress. Let x be the spacing of the joists. *Ans.* $x = 13\frac{1}{3}$ in.

127. Two horizontal steel I-beams are to support the two loads of 3 tons and 4 tons as shown in Fig. 218. (The suspension rods need not be investigated.) The upper load of 3 tons is uniformly distributed over the entire length of the upper beam. The 4-ton load is concentrated on the lower beam at a distance of 3 ft. from the right end. Neglect the weights of the beams.

Compute the maximum bending moment and the maximum total shear for each beam. Then select standard steel beams containing as little metal

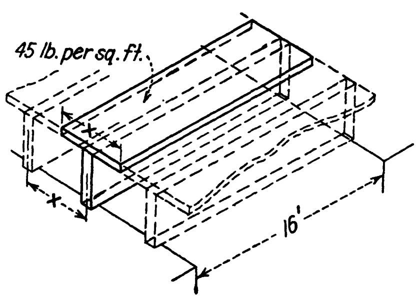


FIG. 217.

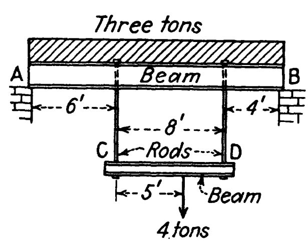


FIG. 218.

as possible, consistent with safety. On account of imperfect lateral support, take 18,000 lb./sq. in. as the maximum safe fiber stress for bending.
Ans. 10-in. 25.4-lb. American Standard for AB and 7 in. 15.3 lb. for CD.

128. Fig. 219. Select a safe, standard, economical steel I-beam to carry the load of 8 tons as sketched in the figure. The load includes the weight of the beam. Use an allowable bending stress of 16,000 lb./sq. in.
Ans. A 6-in. 12.5-lb. American Standard.

129. A 15-in. 50-lb. I-beam (American Standard) is supporting a uniformly distributed load over part of its length (Fig. 220). Z = 64.2 in.^{3} Neglect the weight of the beam. (*a*) Locate position of maximum moment. (*b*)

Find max. moment in terms of W. (c) Maximum safe value of W if S = 18,000 lb./sq. in.

Ans. (a) x = 9.6 ft.; (b) M_m = 7.68W ft.-lb.; (c) W = 12,540 lb.

130. In Problem 129, consider weight of beam and find W (see Note 3, Art. 152). *Ans.* W = 11,250 lb.

131. A 15-in. 50-lb. I-beam (American Standard) is to carry a live load of 430 lb. per foot of length. Beam on end support. Find the maximum safe length of the beam. Consider weight of beam. S = 18,000 lb./sq. in.

Ans. L = 40 ft.

132. A rectangular wooden beam (b not less than $h/2$) is to carry the loads shown in Fig. 221. The wall is 6 in. thick (6 in. ⊥ paper). Select the beam. (The brick wall is assumed to have no beam action.) The weight of the beam may be neglected. Why? S = 1200 lb./sq. in.

Ans. M_m = 3840 ft.-lb. If $b = h/2$, h = 7.72. Take beam 4 by 10 in.

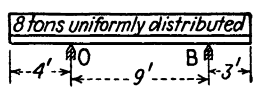


Fig. 219.

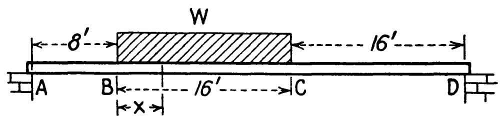


Fig. 220.

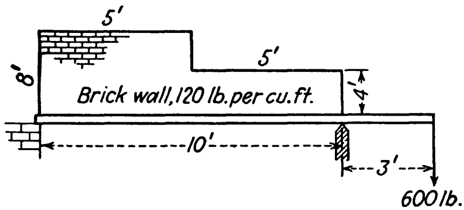


Fig. 221.

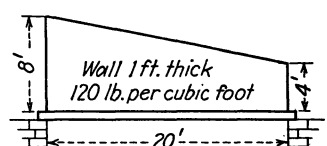


Fig. 222.

133. Select a standard I-beam to carry the load shown in Fig. 222. S = 18,000 lb./sq. in. Neglect weight of beam.

Ans. M_m = 36,000 ft.-lb. 10-in. 25.4-lb. beam.

134. If the weight of the beam of Problem 133 is considered is the beam safe? *Ans.* No.

BEAM UNSYMMETRICALLY LOADED. OBLIQUE LOADING

154. Beams with Two Planes of Symmetry.—In all beam problems so far considered it was assumed that the plane of loading was a plane of symmetry of the beam. That is, it was assumed that the beam was symmetrically loaded.

The flexure formula $M = SI/C$ was derived on the assumption that the neutral axis of a section of the beam is perpendicular to the plane of loading. If the plane of loading is a plane of symmetry of the beam, this assumption will be satisfied (Fig. 189, page 189). Beams, however, may be unsymmetrically loaded.

We shall now consider briefly the case of *unsymmetrical loading*, it being assumed that all assumptions made in Art. 112 or Art. 134, not inconsistent with unsymmetrical loading, are satisfied.

Consider first the simple case of a prismatic beam with two planes of symmetry. For convenience, assume the sections of this beam rectangular. Designate the axes of symmetry of a section as the X- and the Y-axis, respectively (Fig. 223). Consider now any given section of the beam and assume that the plane of loading intersects this section in a line p making any angle θ with the Y-axis as is indicated in Fig. 223. That is, assume that the plane of loading is oblique, making an angle θ with the vertical plane of symmetry. By referring now to Fig. 223, it can be shown that the neutral axis N is not perpendicular to the plane of loading p. Hence the flexure formula is not applicable directly. However, if each load is resolved into two rectangular components, one in the vertical plane and the other in the horizontal plane of symmetry, the flexure formula may be applied to each component separately. For instance, if the y-component of the load is used, then the neutral axis is the X-axis and therefore perpendicular to the plane of the y-component. The stress at a point on a section may be found, then, by adding algebraically the stresses at this point due to the two component loads each in turn acting alone.

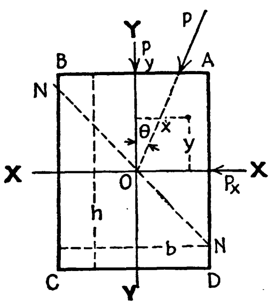


FIG. 223.

Example.—An 8- by 12-in. wooden beam is 10 ft. long and rests on end supports. It carries a load $P = 5000$ lb. at the middle of the beam. The line of action of the load makes an angle of 30° with the vertical plane of symmetry (Fig. 223, $b = 8$ in., $h = 12$ in., $\theta = 30°$).

The axes of symmetry are designated, respectively, as the X-axis and the Y-axis.

Resolve P into its x- and its y-component.

$$P_x = P \sin \theta = 5000 \times 0.50 = 2500 \text{ lb.}$$
$$P_y = P \cos \theta = 5000 \times 0.866 = 4330 \text{ lb.}$$

Stresses Due to P_x.—The neutral axis for P_x is the Y-axis. Hence I must be calculated with respect to the Y-axis.

$$I_y = \tfrac{1}{12} \times 12 \times 8^3 = 512.$$

The maximum bending moment due to P_x (*i.e.*, the bending moment at the middle of the beam with respect to the Y-axis) is

$$M_y = \frac{P_xL}{4} = \frac{2500 \times 10 \times 12}{4} = 75{,}000 \text{ in.-lb.}$$

If S_x equals the stress at a point in the section due to P_x, and c_x equals the distance of this point from the Y-axis (the x-coordinate of the point),

$$S_x = \frac{M_yc_x}{I_y} = \frac{75{,}000 \times c_x}{512} = 146.4c_x.$$

By designating a compressive stress as negative $(-)$, the stresses at the four corners of the middle section (due to P_x) are, respectively,

$$\begin{aligned} (S_x)_A &= -146.4 \times 4 = -585.6 \text{ lb./sq. in.} \\ (S_x)_B &= +585.6. \\ (S_x)_C &= +585.6. \\ (S_x)_D &= -585.6. \end{aligned}$$

Stresses Due to P_y.—In like manner the stresses due to P_y are determined.

$$\begin{aligned} I_x &= \tfrac{1}{12} \times 8 \times 12^3 = 1152. \\ M_x &= \frac{P_yL}{4} = \frac{4330 \times 10 \times 12}{4} = 129{,}900 \text{ in.-lb.} \\ S_y &= \frac{M_xc_y}{I_x} = \frac{129{,}900 \times c_y}{1152} = 112.7c_y. \end{aligned}$$

Therefore

$$\begin{aligned} (S_y)_A &= -112.7 \times 6 = -676.2 \text{ lb./sq. in.} \\ (S_y)_B &= -676.2. \\ (S_y)_C &= +676.2. \\ (S_y)_D &= +676.2. \end{aligned}$$

Combining stress algebraically,

$$\begin{aligned} S_A &= -1261.8 \text{ lb./sq. in.} \\ S_B &= -90.6. \\ S_C &= +1261.8. \\ S_D &= +90.6. \end{aligned}$$

155. Principal Axes.—Let Fig. 224 be a section of a beam of any given shape and let O be any given point in this section. It can be shown that there are always two axes through O such that the moment of inertia of the section with respect to one of

these axes is greater and that with respect to the other axis is less than that with respect to any other line through O. The two axes are called the *principal axes* for the point O. If, then, X and Y are the principal axes for the point O (Fig. 224) and if I_y is greater than I_x, it follows that I_y is greater than I_n and I_x is less than I_n where N is any line through O. It can be shown that (see Art. 156)

I. The two principal axes of a section (for any given point in that section) are always at *right angles* to each other.

II. If $I_x = I_y$ (X and Y being principal axes), then

$$I_x = I_y = I_n;$$

i.e., the moment of inertia is the same for all axes through O. This is a limiting case.

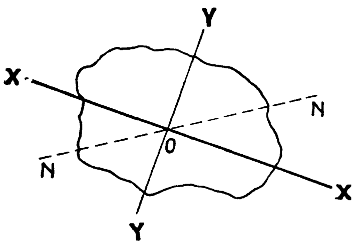


FIG. 224.

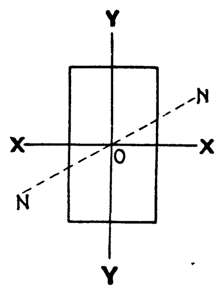


FIG. 225.

III. *An axis of symmetry always is a principal axis for every point on that axis.*

In the discussion that follows, when reference is made to the principal axes of a section, it is to be understood that these are *the principal axes for the centroid* (center of gravity) *of the section; i.e.*, O is the centroid of the section.

Illustration I.—Figure 225 represents a rectangular section. This section has two axes of symmetry, both principal axes. Here I_x is greater than I_y. If N is any line through O, then $I_x > I_n$ and $I_y < I_n$.

Illustration II.—Figure 226 represents a channel section. Axis 1 . . . 1 is an axis of symmetry and hence is a principal axis. Since the principal axes are at right angles to each other, axis 2 . . . 2 is the other principal axis. It follows, therefore, that $I_1 > I_n$ and $I_2 < I_n$. Tables give the values of I_1 and I_2 for the various standard channels.

Illustration III.—Figure 227 represents an angle section with no axis of symmetry. Axis 3 . . . 3 is the axis of least radius of gyration. That is, I_3 is the least moment of inertia of the section. Axis 3 . . . 3 is, therefore, one of the principal axes, the other principal axis being 4 . . . 4. Tables give the values of k_3 (also of I_1 and I_2) for each of the various standard angle sections.

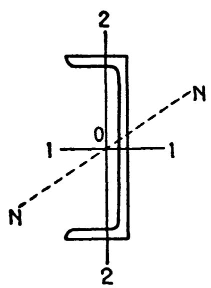


FIG. 226.

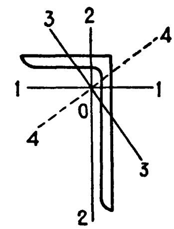


FIG. 227.

156. Proof of Theorems of Previous Article.—Figure 228 represents a section of a beam. For two sets of rectangular axes having the same origin O

$$v = y \cos \theta - x \sin \theta.$$

The moment of inertia of the section with respect to the U-axis is, therefore,

$$I_u = \int v^2 dA$$
$$= \int (y \cos \theta - x \sin \theta)^2 dA$$
$$= \cos^2 \theta \int y^2 dA + \sin^2 \theta \int x^2 dA - 2 \sin \theta \cos \theta \int xy dA. \quad (a)$$

If we let $\int xy dA = P$, Eq. (a) becomes

$$I_u = I_x \cos^2 \theta + I_y \sin^2 \theta - P \sin 2\theta. \quad (b)$$

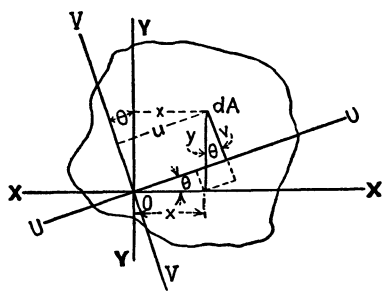


FIG. 228.

Hence, if I_x, I_y, and P have been calculated with respect to any convenient set of rectangular axes (X and Y), I_u is determined for any line U making an angle θ with the X-axis.

To find θ for which I_u is a maximum or a minimum, put $dI_u/d\theta = 0$.

Or [Eq. (*b*)],

$$\frac{dI_u}{d\theta} = -2 \sin \theta \cos \theta \, I_x + 2 \sin \theta \cos \theta \, I_y - 2P \cos 2\theta = 0$$

$$= (I_y - I_x) \sin 2\theta - 2P \cos 2\theta = 0.$$

Or, solving for $\tan 2\theta$,

$$\tan 2\theta = \frac{2P}{I_y - I_x}. \qquad (c)$$

Note that Eq. (*c*) is satisfied for two values of 2θ, 180° apart, *i.e.*, for two values of θ, 90° apart. Hence, for every point O, there are two axes, 90° apart, such that the moment of inertia is a maximum with respect to one of these axes and a minimum with respect to the other. This proves Theorem I (Art. 155); *viz., the two principal axes are always at right angles to each other.*

If X is an axis of symmetry, then, in $\int xy dA$, for every term $x(+y)$ there is a term $x(-y)$ to cancel it so that $\int xy dA = 0$; *i.e.*, $P = 0$. Hence, if X (or Y) is an axis of symmetry, $P = 0$ and Eq. (*b*) becomes

$$I_u = I_x \cos^2 \theta + I_y \sin^2 \theta. \qquad (d)$$

Putting $P = 0$ in Eq. (*c*), $\tan 2\theta = 0$; *i.e.*, $\theta = 0$, or 90°. Hence X is one principal axis and Y is the other. This proves Theorem III; *viz., an axis of symmetry always is a principal axis for any point on that axis.*

Note.—It can be shown that in general, if X and Y are principal axes, $P = 0$. Conversely, if $P = 0$, the axes are principal axes.

Let X and Y be principal axes, and assume that $I_x = I_y = I$. Since $P = 0$, Eq. (*b*) becomes

$$I_u = I (\cos^2 \theta + \sin^2 \theta) = I. \qquad (e)$$

This proves Theorem II, *viz., if $I_x = I_y = I$ (X and Y being principal axes) then the moment of inertia with respect to any axis through O is I.*

157. Shear Center.—Consider now a beam of any given shape. Assume that the loads and reactions all lie in one plane, the load plane. Let the load plane cut a given section of the beam in the line p (Fig. 229). It can be shown that for the given section

there is a *definite* point C (not necessarily the centroid of the section) such that if p goes through C the beam will *not* twist at that section; *i.e.*, no torsional stresses will be induced on that section. The point C is called the *shear center* of that section. In general, *if the plane of loading goes through the shear center of every section of the beam, the beam will not twist.*

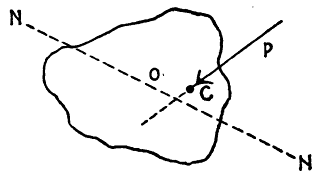


FIG. 229.

Figure 230 represents a channel used as a cantilever. It can be shown that the shear center is at C. If the line of action of P does *not* go through C, the beam will twist as indicated in the figure.

A detailed treatment of the shear center is beyond the scope of this text. For proofs of the statements made, see some recent work on advanced mechanics.*

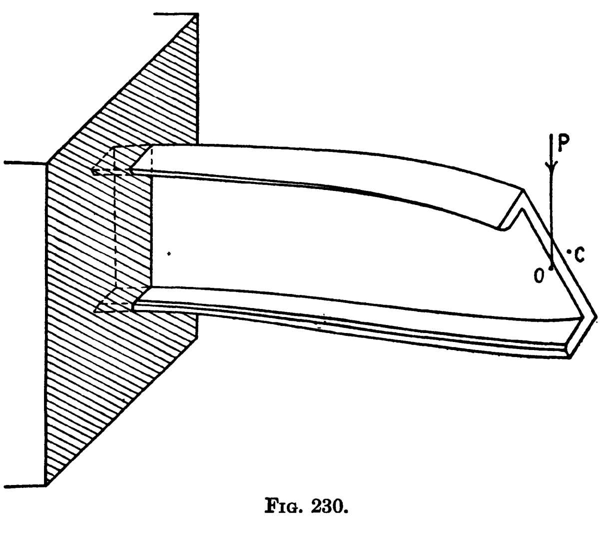


FIG. 230.

The following theorems are very important:

I. *An axis of symmetry of a section always contains the shear center of that section.*

II. Hence *if a section is symmetrical with respect to two axes, the shear center coincides with the centroid of that section.* For

* FÖPPL, "Drang und Zwang," Band II, Sec. 78.
SEELY, F. B., "Advanced Mechanics of Materials."

instance, take an I-beam (Fig. 231). Axes 1 . . . 1 and 2 . . . 2 are axes of symmetry and therefore gravity axes. The shear center, lying on both axes, must coincide with the centroid of that section.

158. Neutral Axis of a Section.—If a beam is unsymmetrically loaded, the neutral axis of a section is not, as a rule, perpendicular to the plane of loading. For instance (Fig. 231), the neutral axis N is not perpendicular to p, the plane of loading.

The following theorems are important:

III. *A neutral axis is always a gravity axis, provided the assumptions of Art.* 134 *are satisfied* (see Art. 138).

IV. *If for a section of a beam the plane of loading is parallel to one of the principal axes* (through the centroid), *then the neutral*

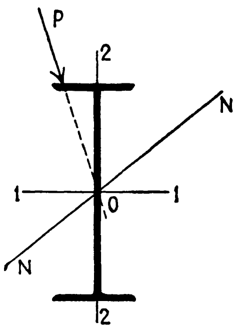


FIG. 231.

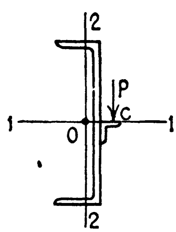


FIG. 232.

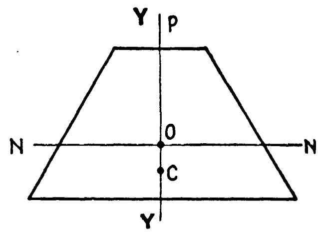


FIG. 233.

axis is the other principal axis. The beam, however, will twist unless the plane of loading goes through the shear center of every section of the beam (see Art. 159).

Illustration I. Channel Section (Fig. 232).—Axis 1 . . . 1 is an axis of symmetry and therefore a principal axis. Hence axis 2 . . . 2 is the other principal axis through the centroid. (The two principal axes always are perpendicular to each other.) The shear center lies on axis 1 . . . 1 (an axis of symmetry always contains the shear center). Let C be the shear center. If then the plane of loading goes through C and is parallel to axis 2 . . . 2, the channel will not twist but will bend about axis 1 . . . 1. Note that *axis* 1 . . . 1 *will be the neutral axis of the section if and only if p is parallel to axis* 2 . . . 2, *and that the beam will not twist if and only if p goes through the shear center C.*

Illustration II. Fig. 233.—Let Y be an axis of symmetry (and therefore a principal axis through the centroid O). The shear

center C lies on Y (an axis of symmetry always contains the shear center). If then the plane of loading is a symmetrical plane (if p coincides with Y), the beam will not twist. Morever, X (the other principal axis through the centroid) will be the neutral axis.

Note.—In all beam problems, unless a statement to the contrary is made, *it is assumed,* for convenience, *that the plane of loading is a symmetrical plane* (Assumption 3, Art. 112), and that, therefore, the neutral axis is a (gravity) axis perpendicular to the plane of loading.

159. A Principal Axis in the Plane of Loading. Neutral Axis.—Figure 234 represents as a free body the part of a beam to the left of the section CD (part A). For convenience, the section is represented as rectangular and the plane of loading

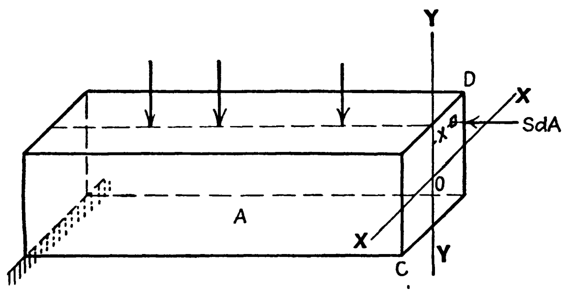


FIG. 234.

is represented as containing the Y-axis. Since equilibrium exists, the internal forces acting on the section CD must hold in equilibrium the external forces acting on A. Hence the sum of the moments with respect to the Y-axis of the internal forces acting on CD must be equal to the sum of the moments with respect to Y of the external forces acting on A. Since the external forces acting on A are parallel to the Y-axis (by assumption), $M_y = 0$. (The moment of a force is zero with respect to a line that is parallel to the line of action of the force.) Therefore, the sum of the moments of the internal forces acting on CD must be zero with respect to the Y-axis. Or, if S is the intensity of stress on an element of area dA distant x from Y,

$$\int x(SdA) = 0. \qquad (a)$$

Figure 235 represents the section CD (of any shape). Let X and Y be the principal axes through the centroid O. For

convenience assume that the plane of loading *contains* the Y-axis (assume that p and Y coincide). Assume that U is the neutral axis. We wish to prove (X and Y being principal axes) that $\alpha = 0$. That is, if the plane of loading contains one principal axis through the centroid (or is parallel to a principal axis), then the neutral axis is the other principal axis.

If U is the neutral axis, then the intensity of stress S acting on an element dA is directly proportional to v, the perpendicular

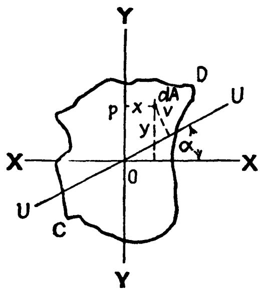


FIG. 235.

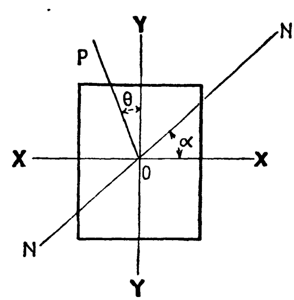


FIG. 236.

distance of the element from the neutral axis U (Art. 135). Hence, if k is a constant,

$$S = kv.$$

Or, in terms of x and y (see Art. 156),

$$S = k(y \cos \alpha - x \sin \alpha).$$

Therefore [Eq. (a)],

$$k\int x(y \cos \alpha - x \sin \alpha)dA = 0$$

or

$$\cos \alpha \int xydA - \sin \alpha \int x^2dA = 0. \qquad (b)$$

Since X and Y are principal axes (by assumption), the first integral is zero ($P = 0$, see note of Art. 156).
Also

$$\int x^2dA = I_y.$$

Therefore

$$I_y \sin \alpha = 0.$$

Or, since I_y is not zero, $\sin \alpha = 0$; *i.e.*, $\alpha = 0$. This proves the theorem; *viz.*, *if the plane of loading contains one principal axis (or is parallel to a principal axis), then the neutral axis is the other principal axis.*

Note.—Figure 236 represents a section (rectangular for convenience). X and Y are the principal axes (axes of symmetry in the figure). The load plane makes an angle θ with the Y-axis. It can be shown that the direction of the neutral axis N is given by the equation

$$\tan \alpha = \frac{I_x}{I_y} \tan \theta. \qquad (c)$$

Note that, if $\theta = 0$, $\alpha = 0$. That is, if p coincides with Y, N will coincide with X (X and Y being principal axes).

Example I.—Let $I_x/I_y = 2$ and $\theta = 30°$ (Fig. 236).

Substituting in Eq. (c), since $\tan 30° = 0.577$,

$$\tan \alpha = 2 \times 0.577 = 1.155.$$

Therefore

$$\alpha = 49°7'.$$

Example II.—Referring to Fig. 231, let the beam be a 15-in. 50-lb. standard I-beam. $I_1/I_2 = 5.47$.

Assume that p makes an angle of 5° with the 2 . . . 2 axis ($\tan 5° = 0.0875$).

$$\tan \alpha = 5.47 \times 0.0875 = 0.479,$$
$$\alpha = 24°37'.$$

160. Oblique Loading. General Case.—If the stresses on a section of a beam subjected to a system of loading are required, it is necessary to proceed as follows:

1. Locate the principal axes through the centroid of the section under consideration. Call the principal axes respectively the X- and the Y-axis.

2. Resolve each force acting on the beam into its x-component and its y-component.

3. Combine algebraically the flexural stress at a point due to the P_x forces acting alone and that due to the P_y forces acting alone. Note that, for the P_x forces, Y is the neutral axis; and that, for the P_y forces, X is the neutral axis (X and Y being principal axes).

Note 1.—If the plane of loading does not go through the shear center of the section, torsional stresses are induced on the section.

Note 2.—Beams in practice frequently are laterally supported in such a way that twisting of the beam is prevented or greatly reduced. In such cases the location of the shear center is a matter of secondary importance.

MOVING LOADS

161. Largest Moment Due to Moving Loads.—In Fig. 237, a simple beam on end supports is shown carrying a system of concentrated loads. When these loads are fixed in magnitude and position, we may compute the reactions, construct the shear diagram, locate the section at which the moment is a maximum, and finally obtain the maximum moment as was done in the examples of Art. 130. Since the loads are concentrated loads, the maximum moment will occur under one of the loads.

Assume now that the loads form a set of moving loads of constant magnitude and at fixed distances apart. This is illustrated in the case of a locomotive running over a deck girder bridge. The pressures of the wheels on the track form a set of moving

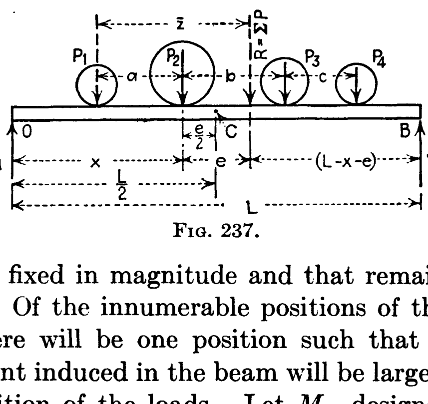


Fig. 237.

loads that are fixed in magnitude and that remain at fixed distances apart. Of the innumerable positions of this set of loads (Fig. 237) there will be one position such that the maximum bending moment induced in the beam will be larger than that for any other position of the loads. Let M_m designate the largest bending moment that will be induced in the beam as the set of moving loads passes over the beam. Evidently, M_m will occur under one of the loads and for a particular position of this load. It is required, therefore, to determine under which load and for which position of this load the largest bending moment M_m will occur. If this load and the position of this load are found, then no greater moment can occur under that load in any other position, and no other load can be found under which a larger moment is possible.

As a first step in the solution of the problem, a rule will be developed for determining the position of the set of moving loads for which the moment under a particular load (P_2, for instance) will be greater than the moment under that load for any other position of the loads.

Let R be the resultant of the loads on the beam. In Fig. 237, $R = P_1 + P_2 + P_3 + P_4 = \Sigma P$. The line of action of this resultant with respect to the load P_1 may be found as follows: Since the moment of the resultant equals the sum of the moments of the several forces,

$$R\bar{z} = P_2 a + P_3(a + b) + P_4(a + b + c),$$

from which $\bar{z}$, the distance of R from P_1 may be calculated.

Let e be the distance of R from P_2, the load under consideration. Also let x equal the distance of P_2 *from the left end of the span.* Putting the summation of moments about B equal to zero,

$$V_1 L - R(L - x - e) = 0 \qquad \text{or} \qquad V_1 = \frac{R(L - x - e)}{L}.$$

Therefore the moment under P_2 is

$$\begin{aligned} M_2 = V_1 x - P_1 a &= \frac{R}{L}(L - x - e)x - P_1 a \\ &= \frac{R}{L}(Lx - x^2 - ex) - P_1 a. \end{aligned}$$

To find the value of x for which M_2 is a maximum, put

$$dM_2/dx = 0,$$

and solve for x. (Note that R, L, a, e, and P_1 are constants.)

$$\frac{dM_2}{dx} = \frac{R}{L}(L - 2x - e) = 0.$$

Therefore

$$L - 2x - e = 0 \qquad \text{or} \qquad \frac{L}{2} = x + \frac{e}{2}.$$

That is, C, the center of the span, bisects the distance between P_2 and R. A similar result will be obtained for any other load, as P_3 or P_4. This leads to the following rule, applicable to a set of moving loads passing over a *simple span* on end supports:

Rule.—*The maximum moment under a given load of a set of moving loads* (as of a group of wheel loads) *occurs when that load is as far on one side of the middle of the span as the resultant of all the loads then on the span is on the other side.*

The rule is applied as follows: *for each load in turn,* place the set of loads so that the moment under that load reaches its maximum value and calculate this moment. The largest of the maximum moments thus obtained is the largest moment that will

be induced in the beam as the set of loads moves over the beam, and this moment (designated by M_m) is the moment to be used in the flexure formula

$$M_m = \frac{SI}{c}.$$

Note 1.—It has been found that, if the load nearest the resultant R is one of the heaviest loads, it will be sufficient to deal only with that load.

Note 2.—In any case, *R always is the resultant of the loads actually on the beam.* For instance, for a particular position of the loads it may happen that the front loads or the rear loads are not on the beam. In that case R must be calculated in amount and in position for the loads actually on the beam. A scale diagram of the set of loads sliding along a scale diagram of the beam is a convenient device for finding what loads are on the beam.

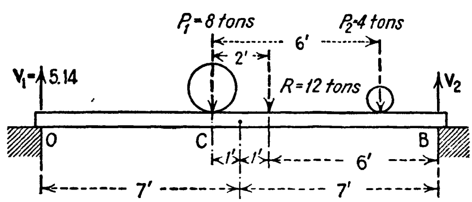


FIG. 238.

Example I. Fig. 238.—Find M_m, the greatest moment induced in the 14-ft. beam as the moving loads pass over the beam.

$$R = \Sigma P = 12 \text{ tons.}$$

$$\bar{z} = \frac{P_2 a}{R} = \frac{4 \times 6}{12} = 2 \text{ ft.} = e.$$

That is, the resultant acts 2 ft. to the right of P_1.

Now place the loads so that P_1 is as far to the left of C (mid-span) as R is to the right of C. With the loads thus placed, M_m will occur under P_1 (see Note 1).

Since the resultant R and its position are known, it is simpler to use the resultant instead of the individual loads when finding V_1.

Putting $M_B = 0$,

$$V_1 \times 14 - 12 \times 6 = 0.$$

Therefore

$$V_1 = 5.14 \text{ tons.}$$

Hence

$$M_1 = V_1 \times 6 = 5.14 \times 6 = 30.8 \text{ ft.-tons} = M_m.$$

Example II. Fig. 239.—A set of three loads on a 30-ft. beam. Find M_m.

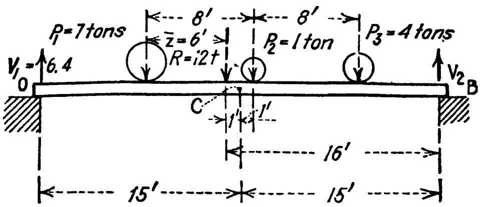


FIG. 239.

$$R = 12 \text{ tons.} \qquad \bar{z} = \frac{4 \times 16 + 1 \times 8}{12} = 6 \text{ ft.}$$

First consider P_2. Place P_2 as far to the right of C as R is to the left of C (Fig. 239).

$$V_1 \times 30 - 12 \times 16 = 0.$$

Therefore

$$V_1 = 6.4 \text{ tons.}$$

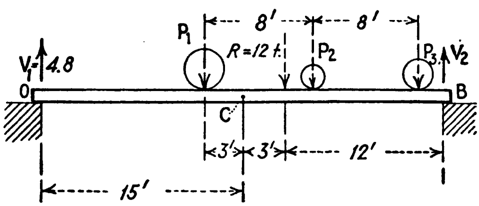


FIG. 240.

Hence

$$M_2 = 6.4 \times 16 - 7 \times 8 = 46.4 \text{ ft.-tons.}$$

Now place P_1 as far to the left of C as R is to the right of C (Fig. 240).

$$V_1 \times 30 - 12 \times 12 = 0.$$

Therefore

$$V_1 = 4.8 \text{ tons.}$$

Hence

$$M_1 = 4.8 \times 12 = 57.6 \text{ ft.-tons} = M_m.$$

Since the load nearest the resultant (*i.e.*, P_2) is not one of the heavier loads, Note 1 cannot be used.

Problem 135. Fig. 241.—The three wheels, at fixed distances apart, roll across the 24-ft. beam. Draw the dimension sketches and compute the maximum moment under each wheel. Design an economical, standard I-beam, considering its own weight. $S = 16{,}000$ lb./sq. in.

Ans. $M_m = 16.45$ ft.-tons.

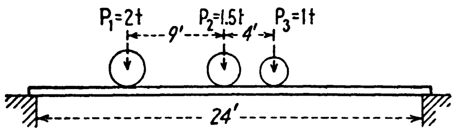


FIG. 241.

SHEARING STRESS IN A BEAM

162. Given a loaded beam (Fig. 242*a*). Let M equal the bending moment at the section CD of the beam. In Art. 139, it is shown that

$$M = \frac{SI}{c}.$$

Or, solving for S,

$$S = \frac{Mc}{I}. \tag{a}$$

In Eq. (*a*), S equals the stress in the *outer* fiber at the given section, I equals the moment of inertia of the section (with respect

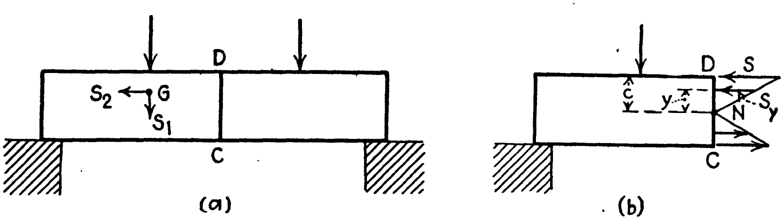


FIG. 242.

to the neutral axis N, Fig. 242*b*), and c equals the distance of the outer fiber from the neutral axis. Moreover, the intensity of the normal stress in a fiber distant y from the neutral axis is (Art. 135),

$$S_y = \frac{y}{c}S = \frac{y}{c}\frac{Mc}{I} = \frac{My}{I}. \tag{b}$$

By means of Eq. (*b*) the intensity of the normal stress at any given point in a section can be found if M and I are known. Thus, if the bending moment at a section is $M = 1{,}050{,}000$ in.-lb. and if $I = 525$ in.^{4}, then the stress at a point 4 in. from the neutral axis ($y = 4$ in.) is

$$S_4 = \frac{My}{I} = \frac{1{,}050{,}000 \times 4}{525} = 8000 \text{ lb./sq. in.}$$

It is now desired to develop a formula that will give the intensity of the shearing stress at a point in a beam (either on a horizontal or on a vertical plane). Before developing this formula, reconsider the theorem which states that the intensities of the shearing stress on the four faces of an elemental prism are always equal (Art. 19). That is (Fig. 243), consider an elemental cube upon whose faces (faces perpendicular to plane of paper) act shearing stresses of intensities S_1, S_2, S_3, and S_4, respectively. Putting Σ moments about $E = 0$,

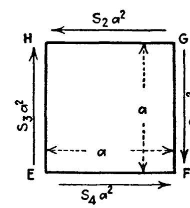


FIG. 243.

$$(S_1a^2)a - (S_2a^2)a = 0.$$

Therefore

$$S_1 = S_2.$$

Now G may be any given point in a body, and GH and GF may be thought of as two mutually perpendicular planes. Hence the theorem may be stated as follows:

At any given point in a body, the intensities of the shearing stress on planes at right angles to each other are always equal.

This theorem will be used repeatedly in the following form:

At a point in a beam, the intensity of the shear on a horizontal plane equals the intensity of the shear on a vertical plane.

163. Shearing Stress in a Beam.—In Fig. (244*a*), BD and $B'D'$ are two sections a distance dx apart, and EE' is a horizontal plane distant y' from the neutral surface NN'. Take free the block $DEE'D'$ (Fig. 244*b*, side view). Let H equal the sum of normal forces acting on the left face of this block, and H' equal that acting on the right face. As a rule H' does not equal H. Hence, as a rule, a shearing stress whose intensity will be designated by S_s is induced on EE' the lower face of this block. If t is the thickness of the beam at E (or E'), and is therefore the

dimension ($\perp$ the plane of the paper) of the lower face of the block, then the area of this face is tdx, and the shearing force on the area is $S_s tdx$.

Since the block is in equilibrium, $\Sigma F_x = 0$.

$$H' - H - S_s tdx = 0.$$

Or

$$S_s tdx = H' - H. \tag{a}$$

For convenience, the cross-section of the beam is represented as rectangular. With reference to Fig. 244a, if dA is the element of area (dark shaded area) at a distance y from the neutral axis N, and S_y is the intensity of the normal stress on this area, then the

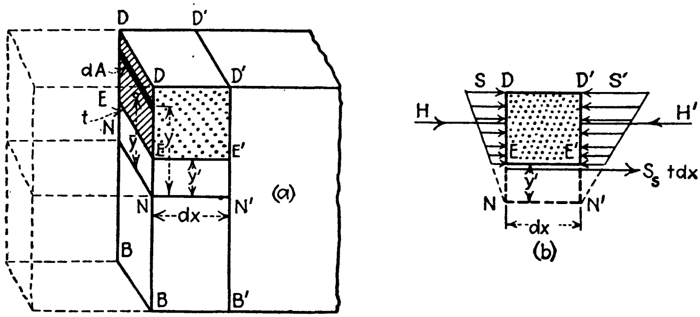


FIG. 244.

total force acting on the left face of the block (line shaded area) is

$$H = \int_{y'}^{c} S_y dA.$$

Or, since $S_y = My/I$ [Art. 162, Eq. (b)],

$$H = \frac{M}{I}\int_{y'}^{c} ydA, \tag{b}$$

where M = bending moment at the section BD.
In like manner,

$$H' = \int_{y'}^{c} S'_y dA = \frac{M'}{I}\int_{y'}^{c} ydA. \tag{c}$$

where M' = bending moment at the section $B'D'$.

Substituting the values of H and H' as given by Eqs. (b) and (c) into Eq. (a),

$$S_s tdx = \frac{M' - M}{I}\int_{y'}^{c} ydA.$$

Or, since $M' - M = dM$ and is the increase in bending moment as we go from section BD to section $B'D'$ (BD and $B'D'$ being an infinitesimal distance dx apart),

$$S_s t dx = \frac{dM}{I}\int_{y'}^{c} y dA.$$

Or, solving for S_s,

$$S_s = \frac{dM}{dx}\frac{1}{tI}\int_{y'}^{c} y dA. \qquad (d)$$

From Elementary Mechanics (Art. 137),

$$\int_{y'}^{c} y dA = A'\bar{y}$$

where A' = area of section above EE (line shaded area, Fig. 244a).
$\bar{y}$ = distance of the center of gravity of A' from the neutral axis N.

Also (Art. 124)

$$\frac{dM}{dx} = V = \text{vertical shear at the section } BD.$$

Hence Eq. (d) may be written

$$S_s = \frac{V}{tI}\int_{y'}^{c} y dA = \frac{V}{tI}A'\bar{y}.^* \qquad (11)$$

In Eq. (11),

S_s = intensity of the shearing stress on a horizontal plane and therefore *also on a vertical plane* at a point E distant y' from the neutral surface.
V = total vertical shear at the section through that point.
I = moment of inertia of that section.
t = thickness of beam (dimension perpendicular to plane of paper) at that point.
A' = area of section above that point.
$\bar{y}$ = distance of center of gravity of A' from the neutral axis of that section.

Example. Fig. 245.—The vertical shear at a section of an 8- by 12-in. rectangular beam is $V = 6400$ lb. Find the intensity of the shear on this section (a) at the neutral axis; (b) at a point 2 in. above the neutral axis.

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12} \times 8 \times 12^3 = 1152 \text{ in.}^4$$
$$t = 8 \text{ in. for both cases.}$$

* Some writers designate the product $A'\bar{y}$ the static moment.

(*a*) A' (shaded area, Fig. 245*a*) $= 8 \times 6 = 48$ sq. in., $\bar{y} = 3$ in. Therefore

$$S_s = \frac{V}{tI}A'\bar{y} = \frac{6400}{8 \times 1152} \times 48 \times 3$$
$$= 100 \text{ lb./sq. in. at the neutral axis } NN.$$

(*b*) A' (shaded area, Fig. 245*b*) $= 8 \times 4 = 32$ sq. in., $\bar{y} = 4$ in. Therefore

$$S_s = \frac{6400}{8 \times 1152} \times 32 \times 4$$
$$= 88.9 \text{ lb./sq. in. at a point 2 in. above } NN.$$

Problem 136.—A 4- by 8-in. rectangular wooden beam is 12 ft. long and carries a central concentrated load of $P = 2$ tons. Find the maximum unit shear induced in the beam. *Ans.* $S_s = 93.75$ lb./sq. in.

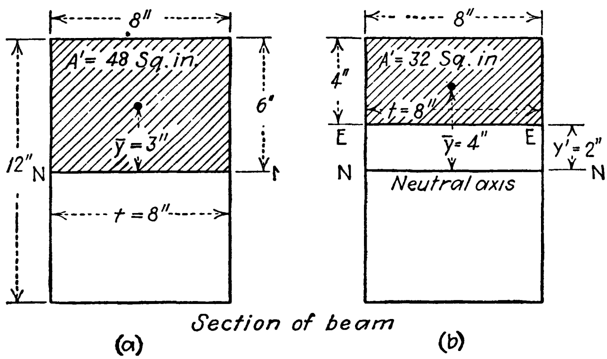


FIG. 245.

164. Mode of Distribution of the Shear on a Vertical Section of a Loaded Beam.—Figure 246 represents a vertical section of a loaded beam. If V is the total shear on that section, then the intensity of the shear at a point distant y' from the neutral axis is [Eq. (11)]

$$S_s = \frac{V}{tI}\bar{y}A'. \qquad (a)$$

Evidently, $A' = 0$ for an outer element. That is, when E and D coincide, A' (shaded area) $= 0$, and therefore $S_s = 0$. Moreover, it is found that, for beams of ordinary shapes, S_s usually is a maximum at the neutral axis.* Hence *the intensity of shear*

* It is possible to devise a section (Fig. 247) such that S_s is not a maximum at the neutral axis. This is also the case for a square with a diagonal set horizontally (Fig. 248).

on a section of a beam is zero at an outer element and, as a rule, is a maximum at the neutral axis.

Rectangular Section. Fig. 249.—For a rectangular section, $t = b$ and $dA = bdy$. Hence the unit shear at a distance y'

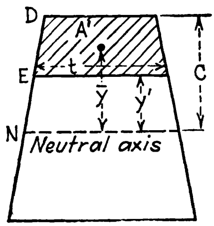


FIG. 246.

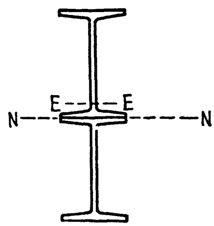


FIG. 247.

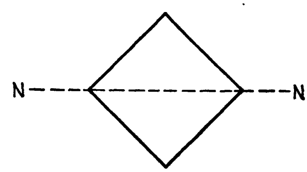


FIG. 248.

from the neutral axis in a rectangular beam is

$$S_s = \frac{V}{Ib}\int_{y'}^{c} ybdy = \frac{V}{I}\left(\frac{c^2}{2} - \frac{y'^2}{2}\right). \tag{b}$$

Equation (b), considered as an equation between S_s and y', is the equation of a parabola. Hence the shearing stress on a rectangular section is parabolically distributed as shown in Fig. 249b.

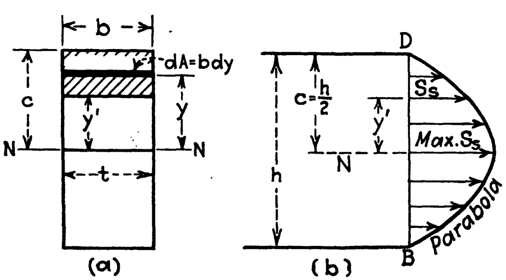


FIG. 249.

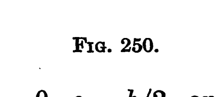


FIG. 250.

At the neutral axis, $y' = 0$. Putting $y' = 0$, $c = h/2$, and $I = \frac{1}{12}bh^3$ in Eq. (b),

$$\text{Max. } S_s = \frac{3}{2}\frac{V}{bh} = \frac{3}{2} \times (\text{total shear} \div \text{area of section}) \tag{12}$$

$$= \frac{3}{2} \times \text{average unit shear on that section.}$$

For instance, if the total shear on an 8- by 12-in. section is $V = 6400$ lb., then the shear at the neutral axis (maximum unit shear) is

$$\text{Max. } S_s = \frac{3}{2} \times \frac{6400}{8 \times 12} = 100 \text{ lb./sq. in.}$$

Equation (12) may be derived directly from Eq. (*a*), *i.e.*, from

$$S_s = \frac{V}{tI}\bar{y}A'.$$

Thus, for the neutral axis, $A' = bh/2$, and $\bar{y} = h/4$. Also, $t = b$ and $I = bh^3/12$.
Making these substitutions in Eq. (*a*),

$$S_s = \frac{3}{2}\frac{V}{bh}.$$

Solid Circular Section.—For a solid circular section of radius r, the unit shearing stress at the neutral axis is

$$S_s = \frac{4}{3}\frac{V}{\pi r^2} = \frac{4}{3} \times \text{average unit shear on the section.* } \quad (13)$$

I-beam Sections.—The various commercial I-beams are not similar and no general relation exists between the dimensions of the section. Accordingly, no general expression or formula similar to formula (12) for rectangular sections or to formula (13) for circular sections can be found for I-beams.

Consider the beam whose section is given in Fig. 251. The value of S_s at a point distant y' from the neutral axis is

$$S_s = \frac{V}{tI}A'\bar{y} = \frac{V}{tI}(A_1\bar{y}_1 + 2A_2\bar{y}_2 + 2A_3\bar{y}_3)$$

where t = thickness of web at the point E at which the unit shear is to be found.

If the values of S_s are found for the various points in the section and these values are plotted, the stress figure takes the form indicated in Fig. 251*b*.

Example. Fig. 252.—A 12-in. 28-lb. WF beam has the dimensions indicated in the figure. $I = 213.5$ in.^{4}. Required to find maximum S_s in terms of the vertical shear V.

Maximum S_s will occur at the neutral axis. Hence, by considering the upper half of the section,

* For a semicircle of radius r,

$$\bar{y} = \frac{4r}{3\pi}, \; A' = \frac{\pi r^2}{2}, \; t = 2r, \text{ and } I = \frac{1}{4}\pi r^4.$$

Substitute in Eq. (*a*).

$$\text{Max. } S_s = \frac{V}{0.24 \times 213.5}[6 \times 0.24 \times 3 + (6.50 - 0.24) \times 0.42 \times (6 - 0.21)]$$
$$= 0.383V.$$

165. Approximate Value of Maximum S_s in an I-beam.—In practice it is customary to assume that the maximum unit shear on a section of an I-beam is the same as if the total shear V at that section were distributed uniformly *over the web*, the web being considered as extending through the flanges. That is, it is assumed that, for an I-beam,

$$\text{Max. } S_s = \frac{V}{th} \tag{14}$$

where t = thickness of web.
h = height of beam.

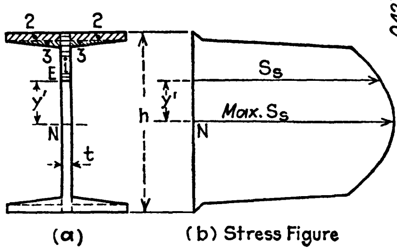


FIG. 251.

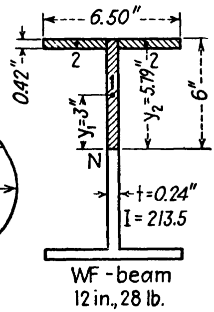


FIG. 252.

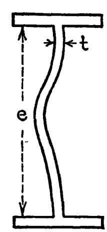


FIG. 253.

Example I.—Using Eq. (14), find maximum S_s for the WF beam given in Fig. 252.

$$\text{Max. } S_s = \frac{V}{0.24 \times 12} = 0.347V.$$

The exact value of maximum S_s for this beam is $0.383V$. The approximate value is about 9.5 per cent too small.

Example II.—If V equals the total shear on a section of a 15-in. 60.8-lb. standard I-beam, find maximum S_s. From Table III, Appendix, $t = 0.59$ in.

$$\text{Max. } S_s = \frac{V}{0.59 \times 15} = 0.113V.$$

The exact value is $0.133V$. The approximate value is about 15 per cent too small.

Equation (14) was applied to a wide range of rolled I-beams and gave results in error from 5 to 22 per cent. *If, therefore, the unit shear in an I-beam must be determined accurately, the exact formula [Eq. (11)] should be used* and not the approximate formula [Eq. (14)].

Note.—Corresponding to a safe flexural stress of

$$S = 18{,}000 \text{ lb./sq. in.},$$

the safe shearing stress [as determined by Eq. (14)] usually is taken as $S_s = 12{,}000$ lb./sq. in. Shear in itself, however, is of little importance in an I-beam. If the shear is excessive the web will fail in buckling rather than in shear (Fig. 253). As the result of experiments it has been found that girders will be safe against *buckling of the web* if the unit shear [as determined by the approximate formula, Eq. (14)] does not exceed 12,000 lb./sq. in. provided h/t does not exceed 70 where h equals the distance between flanges (or flange angles in a built-up beam) and t equals the thickness of web. When h/t is greater than 70, the allowable shear must be decreased.*

166. Design of Beams for Flexure and Shear.—The most convenient method of procedure, as a rule, is as follows: *First design for flexure and then test for shear.*

Example I. Fig. 254.—Design a wooden beam to carry the loading shown. Width of beam is to be not less than $\frac{1}{3}$ its height. Take

$$S' = 1000 \text{ lb./sq. in.} \qquad \text{and} \qquad S'_s = 100 \text{ lb./sq. in.}$$

$$M_m = M_O = 1000 \times 3 + 3000 \times 1.5 = 7500 \text{ ft.-lb.} = 90{,}000 \text{ in.-lb.}$$

$$90{,}000 = 1000\frac{I}{c}.$$

Therefore

$$\frac{I}{c} = 90 \text{ in.}^3$$

If $b = \frac{1}{3}h$,

$$\frac{I}{c} = \frac{1}{6}bh^2 = \frac{1}{18}h^3.$$

* See "A.I.S.C. Manual," Safe Loads for Beams, explanatory notes.

Therefore

$$90 = \frac{1}{18}h^3 \qquad \text{or} \qquad h = 11.75 \text{ in.}$$

Nearest commercial size is a 4- by 12-in. beam. Area of section = 48 sq. in.

Now test for shear. $V_m = 4000$ lb. Therefore [Eq. (12), Art. 164], letting A equal the area of section required for shear,

$$100 = \frac{3}{2} \times \frac{4000}{A} \qquad \text{or} \qquad A = 60 \text{ sq. in.}$$

The area required for shear is greater than the area of the section selected (60 > 48). Hence a 4- by 12-in. beam is unsatisfactory. Select a 6- by 12-in. beam. Area of section = 72 sq. in. > 60.

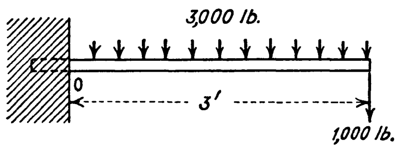


FIG. 254.

Since the I/c of this beam is greater than 90 (that needed for flexure), the second beam is satisfactory.

Example II.—In Example III of Art. 153, the beam selected on the basis of flexure is an 18-in. 47-lb. WF. Test this beam for shear.

$$V_m = 14 \text{ tons.} \qquad S'_s = 12{,}000 \text{ lb./sq. in.}$$

From Table II, Appendix, $t = 0.52$ in. and $h = 17.9$ in. Therefore area of web $= th = 9.30$ sq. in. If A equals the area of web required for shear [Eq. (14), Art. 165],

$$S'_s = \frac{V_m}{A} \qquad \text{or} \qquad 12{,}000 = \frac{14 \times 2000}{A}.$$

Therefore

$$A = 2.33 \text{ sq. in.} < 9.30.$$

Note.—It is seldom that shear governs in an I-beam. Frequently, however, deflection governs. That is, a limitation is placed upon the amount the beam may deflect. Deflection will be considered in a later chapter.

PROBLEMS

137. What should be the height of a wooden floor joist 3 in. wide and 15 ft. long if it is to carry a load of 3200 lb. uniformly distributed? Consider shear at 100 lb./sq. in. and bending stress at 1200 lb./sq. in. *Ans.* 12 in.

138. A 24-in. 110-lb. American Standard I-beam is 7 ft. long. What is the maximum safe load uniformly distributed that this beam can carry? Take $S = 18{,}000$ lb./sq. in. and $S_s = 12{,}000$ lb./sq. in. End supports. *Ans.* $W = 388{,}000$ lb.

167. I-beam on End Supports. Span Limit. *Case* I. *Beam Uniformly Loaded.*—The equation of safe loading is

$$\frac{1}{8}WL = S'\frac{I}{c}$$

where S' = safe unit stress in flexure.
For a given I-beam, I/c is constant.
Therefore

$$WL = 8S'\frac{I}{c} = \text{constant} = k \text{ (say)} \tag{a}$$

or

$$W = \frac{k}{L}.$$

Hence, for a given I-beam, as far as flexure is concerned, W may be increased if L is decreased. Shear, however, imposes a limitation upon the maximum value W may have. If S_s' equals the safe unit shearing stress,

$$S_s' = \frac{V_m}{th}.$$

Or, since $V_m = W/2$,

$$W = 2S_s'th. \tag{b}$$

Note that Eq. (*b*) does not contain L. Hence the maximum safe load W the beam can carry in shear is independent of the length of the beam.

By substituting the value of W as determined by Eq. (*b*) into Eq. (*a*) and solving for L, a length L_0 is determined such that the shearing stress in the beam reaches its maximum allowable value S_s' at the same time that the flexural stress reaches its maximum allowable value S'. Moreover, if $L < L_0$, shear governs [Eq. (*b*) must be used to determine W]; and if $L > L_0$, flexure governs [Eq. (*a*) must be used]. For convenience, L_0

is called the *span limit* for a uniformly loaded I-beam (also for a channel) on end supports.

*Note.**—Handbook tables give the span limit for a uniformly loaded beam (I-beam or channel). If the length of the beam is less than that designated as the span limit, shear governs and W, the load the beam can safely carry, is determined by Eq. (b). In these tables,

$$S' = 18{,}000 \text{ lb./sq. in.}; \qquad \text{and} \qquad S'_s = 12{,}000 \text{ lb./sq. in.},$$

provided $h/t < 70$,

where h = distance between flanges.

t = thickness of web.

If $h/t > 70$, the value of S'_s must be decreased.

Case II. *Beam Loaded Centrally.*—The equation of safe loading is

$$\frac{1}{4}PL = S'\frac{I}{c} \qquad \text{or} \qquad PL = 4S'\frac{I}{c}.$$

Also, since

$$S'_s = \frac{V_m}{th} = \frac{P}{2th},$$

$$P = 2S'_s th.$$

By comparing with Eqs. (a) and (b), it is seen that the span limit for a centrally loaded beam on end supports is one-half that for this beam uniformly loaded. For instance, if $L_O = 6$ ft. for a beam uniformly loaded, then $L_O = 3$ ft. for this beam centrally loaded.

Example I.—A 12-in. 28-lb. WF is to be used as a simple beam on end supports. For this beam $h = 12$ in., $t = 0.24$ in., and $I/c = 35.6$ in.^{3}. By taking

$$S' = 18{,}000 \text{ lb./sq. in.}$$

and $S'_s = 12{,}000$ lb./sq. in., it is required to find L_O, the span limit: (a) beam uniformly loaded; (b) beam centrally loaded.

(a) For safety in flexure [Eq. (a)], beam uniformly loaded,

$$\frac{WL}{8} = 18{,}000 \times 35.6 \qquad \text{or} \qquad WL = 5{,}126{,}000. \qquad (c)$$

Note that, in Eq. (c), L must be expressed in inches.

For safety in shear [Eq. (b)],

$$12{,}000 = \frac{W}{2 \times 0.24 \times 12} \qquad \text{or} \qquad W = 69{,}120 \text{ lb.} \qquad (d)$$

* See "A.I.S.C. Manual."

Solving Eqs. (*c*) and (*d*),

$$L_O = 74.2 \text{ in.} = 6.18 \text{ ft. (beam uniformly loaded).}$$

(*b*). If loaded centrally,

$$L_O = 3.09 \text{ ft. (central concentrated load).}$$

Example II.—In Example I, S'_s is given as 12,000 lb./sq. in. This assumes that $h/t \gtreqless 70$. Find h/t for the given beam. Flange thickness = 0.420 in., t = 0.240 in. (see Table II, Appendix). Hence

$$h = 12 - 2 \times (0.420) = 11.16$$

Therefore

$$\frac{h}{t} = \frac{11.16}{0.240} = 46.5 < 70.$$

168. Rectangular Beams on End Supports. Span Limit. *Case* I. *Uniformly Loaded.*—For a rectangular beam $I/c = \frac{1}{6}bh^2$. Therefore

$$\frac{1}{8}WL = S'\frac{I}{c} = \frac{1}{6}S'bh^2 \qquad \text{or} \qquad WL = \frac{4}{3}S'bh^2. \qquad (a)$$

If S'_s equals the safe unit stress in shear [Art. 164, Eq. (12)],

$$S'_s = \frac{3}{2}\frac{V_m}{bh} = \frac{3}{4}\frac{W}{bh}\left(\text{since } V_m = \frac{W}{2}\right) \qquad \text{or} \qquad W = \frac{4}{3}S'_sbh. \qquad (b)$$

Dividing Eq. (*a*) by Eq. (*b*), member by member, we obtain as the *span limit*

$$L_O = \frac{S'}{S'_s}h. \qquad (c)$$

Case II.—In like manner, for a central concentrated load, the *span limit* is

$$L_O = \frac{S'}{S'_s}\frac{h}{2}.$$

Wooden Beams.—For wood in general, *kind not specified,* we may take $S'/S'_s = 12$. Thus, if $S' = 1200$ lb./sq. in., we may take $S'_s = 100$ lb./sq. in. Putting $S'/S'_s = 12$ (h and L_O in inches),

$$L_O = 12h \qquad \text{or} \qquad h = \frac{L_O}{12}, \text{ beam uniformly loaded.}$$

$$L_O = 6h \qquad \text{or} \qquad \frac{h}{2} = \frac{L_O}{12}, \text{ concentrated load at the middle.}$$

This leads to the following convenient rule: *A wooden beam on end supports should be tested for shear* (1) if its height in inches is greater than its length in feet (beam uniformly loaded); (2) if half its height in inches is greater than its length in feet (concentrated load at the middle).

Example I.—A rectangular wooden beam on end supports is to be uniformly loaded to its full capacity. Beam is 8 ft. in length, and 10 in. in height. Is it necessary to test for shear?

Yes $(10 > 8)$.

Note.—This means that shear governs. Equation (*b*) should be used to determine W.

Example II.—If the beam of Example I is to be loaded centrally, is it necessary to test for shear? No. $(5 < 8)$.

Note.—This means that flexure governs. The flexure formula should be used to determine the safe load.

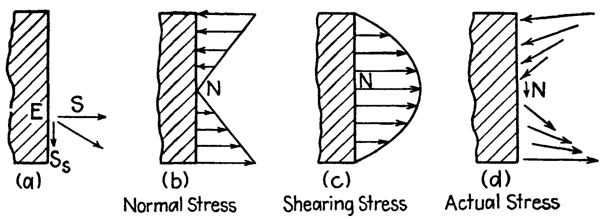


FIG. 255.

169. Actual Distribution of the Stress on the Section of a Beam.—Assume that, at a point E in a section (Fig. 255*a*), S equals the intensity of the normal stress and S_s equals that of the shearing stress. If the two stress intensities are combined, their resultant is the actual stress intensity at that point and acts obliquely to the section.

For convenience, consider a rectangular beam. The normal stress figure consists of two triangles (Fig. 255*b*). The shearing stress figure is parabolic (Fig. 255*c*).* Note that the shearing stress on the outer fibers is zero (Fig. 255*c*). Hence on the outer fibers the resultant stress acts wholly normally, as is shown in Fig. 255*d*. On the other hand, the normal stress is zero at the neutral axis (Fig. 255*b*) so that the resultant stress acts wholly tangentially at the neutral axis. At any other point on the section, the resultant stress acts obliquely. Figure 255*d* is, therefore, the actual stress figure for that section.

* The shearing stress acts along the section but for convenience is represented by lines normal to the section.

170. Effect of Shear on a Section of a Beam. Fig. 256.—Figure 256*a* represents a portion of a beam before the beam is loaded (side view). Let *BD* be a plane section. Consider the element *A*. Before the beam is loaded, the angles at the corners of this element are all right angles as is indicated in Fig. 256*a*.

Assume now that the beam is loaded. If there is a shearing stress on the section *BD*, there will be a shearing stress on the right face of the element *A* and hence (Art. 19) there must be shearing stresses on all four faces of this element as is indicated in Fig. 256*b*. As a result, the angles at the corners of this element are no longer right angles. The shearing stress, however, is zero on an outer element and a maximum on an element at the neutral axis. Elements along the section, therefore, are une-

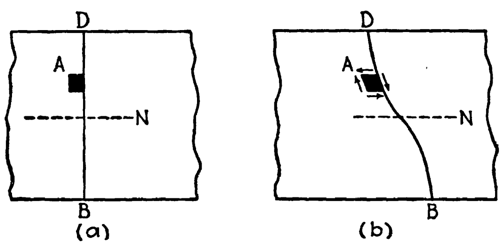


FIG. 256.

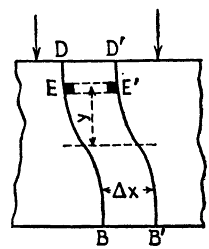


FIG. 257.

qually distorted and the section becomes curved (warped) as is indicated in Fig. 256*b*. In general, if shear acts on a section, that section becomes curved.*

Consider now two sections originally plane and parallel and at a distance Δx apart. If the beam carries *no* load between *D* and *D'* (Fig. 257), then $V = V'$. That is, with no load between *D* and *D'*, the vertical shear at *D* equals that at *D'* and hence the angular distortion of an element along *BD* equals that of the corresponding element along *B'D'*. Accordingly, the curving of the two sections as due to equal shearing stresses acting on them will have no effect upon the length of a fiber such as *EE'*. Any change in the length of this fiber must be due then to the bending moment in the beam. (The effect of bending is not shown in the figure.)

* The angular distortion of an element is very small as a rule. For instance, under usual conditions of loading, the curving of a section of a bent rubber beam is hardly perceptible to the naked eye.

If the beam carries a uniformly distributed load between D and D', the vertical shears at the two sections differ by an amount

$$\Delta V = V' - V = w\Delta x,$$

where w = load per unit length of beam.

It can be shown, however, that for beams of usual proportions this difference between the vertical shears at the two sections has no appreciable effect upon the length of a fiber such as EE'.

In general, *for beams of usual proportions and under usual conditions of loading, no appreciable error is made in assuming that the longitudinal strain of a fiber is due solely to the bending moment in the beam.*

The derivation of the flexure formula is based upon the assumption that, within the elastic limit, plane sections remain plane sections during bending. The "assumption of plane sections" is, however, merely a matter of convenience since from it follows the essential fact that the longitudinal strain of a fiber is directly proportional to the distance of this fiber from the neutral surface. This may be expressed in another way. It is assumed that *the longitudinal strain of a fiber is due solely to the bending moment in the beam and that this strain is the same as if plane sections remained plane sections during bending.*

Note.—The "common theory of flexure" is based upon assumptions that are more or less approximations. Elaborate investigations, both mathematical and experimental, show, however, that for beams of usual proportions and under usual conditions of loading the results obtained from the common theory are in very fair agreement with facts.

STRESS BEYOND THE ELASTIC LIMIT. MODULUS OF RUPTURE

171. Distribution of Fiber Stress When the Elastic Limit Is Exceeded.* *Case* I.—For some materials, such as steel, the elastic limit in tension is the same as that in compression. If then a symmetrical steel beam (an I-beam, say) is overstressed (stressed beyond the elastic limit), corresponding fibers on the two sides of the neutral axis will be overstressed the same amount. If in Fig. 258 it is assumed that fibers up to E and down to E' are

* For most engineering materials, the elastic limit and the proportional limit are practically the same. When this is not the case, the proportional limit is meant.

not overstressed, then EE' is a straight line. Beyond E (or E') fibers are overstressed. Now the stress in an overstressed fiber does not increase as fast as the strain. Thus (Fig. 46) the ratio of stress to strain becomes smaller as the prism is overstressed. For instance, in the neighborhood of the yield-point, the strain is comparatively large with little or no increase in the stress. Accordingly, the line EC (or $E'C'$) of the stress figure (Fig. 258) becomes curved, indicating that the stress in a fiber above E (or below E') is not proportional to the distance of this fiber from the neutral axis.

Case II.—For some materials, the elastic limit in tension is not the same as that in compression. For wood, the elastic limit in tension is greater than that in compression. Since wooden

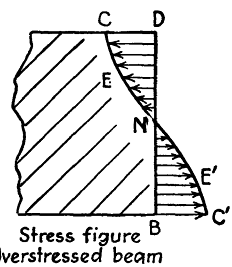


FIG. 258.

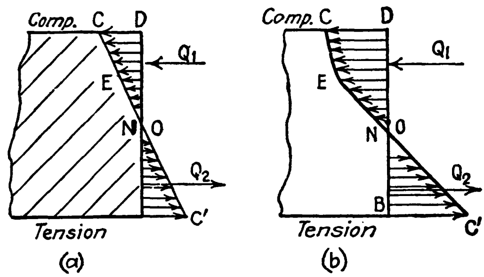


FIG. 259.

beams are rectangular as a rule, consider a rectangular wooden beam. So long as the elastic limit is not exceeded, the stress figure takes the form indicated in Fig. 259*a*, where N, the neutral axis, coincides with O, the gravity axis (Art. 138).

Assume now that the elastic limit in the upper side of the beam (compression side) is exceeded but that the elastic limit in the lower side (tension side) is not exceeded. The stress figure takes the form indicated in Fig. 259*b*. Note that N, the neutral axis, does not coincide with O, the gravity axis. The neutral axis is shown as having shifted toward the side that is not overstressed. It is readily seen why this should be the case. Since equilibrium exists, Q_1 must equal Q_2. That is, the total force Q_1 above the neutral axis must be equal and opposite to the total force Q_2 below the neutral axis. As long as NEC (Fig. 259*a*) is a straight line, N and O coincide. When NEC begins to curve (Fig. 259*b*), N must shift down to keep Q_1 equal to Q_2.

Note.—For a rectangular section, Q_1 is proportional to the area $NECD$ and Q_2 is proportional to the area NBC'. Hence, when EC becomes curved, N must shift down to keep the two areas equal.

In general, when a beam is overstressed, the neutral axis shifts toward that side of the beam that is less overstressed.

172. Section Unsymmetrical with Respect to the Horizontal Gravity Axis. Fig. 260.—Let the stress in the outer fibers at D and at B be S_1 and S_2, respectively. Within the elastic limit (strictly, within the proportional limit),

$$\frac{S_1}{S_2} = \frac{c_1}{c_2}. \tag{a}$$

Evidently, it will be advantageous (as far as strength in flexure is concerned) if the beam is so shaped that the elastic limit (or the allowable stress) in tension and that in compression are reached at the same time. That is [Eq. (a)], if S_1 is the elastic limit for the fibers at D, and S_2 is that for the fibers at B, it will be advantageous (as far as strength in flexure is concerned) so to shape the section that

$$\frac{S_1}{S_2} = \frac{c_1}{c_2}.$$

Note.—It must be remembered that cost frequently is the determining factor. If a wooden beam is used, the extra cost involved in shaping the beam as suggested above makes such a beam, as a rule, not economical. It is usually more economical to use beams of standard shapes.

Illustration.—For cast iron, the stress-strain relation from the start is not the same for compression as it is for tension. Moreover, there is no well-defined proportional limit for cast iron in tension. Referring to Fig. 50, note that the stress-strain diagram for compression is very nearly a straight line up to 18,000 lb./sq. in. and that the stress-strain diagram for tension crosses that for compression at 9000 lb./sq. in. If the beam is so shaped that, when the tension side reaches 9000 lb./sq. in. at the same time that the compression side reaches not more than 18,000 lb./sq. in., the shifting of the neutral axis is not excessive and within these limits no great error is made in assuming that the stress-strain relation for tension is the same as that for compression.

Formerly, cast-iron beams were used extensively and they were so shaped (Fig. 261) that $c_1 = 2c_2$, *i.e.*, so that the outer fibers on the compression side were twice as far from the gravity axis as the outer fibers on the tension side.

173. Modulus of Rupture.—For convenience, consider a beam symmetrical with respect to the horizontal gravity axis. Assume that for this beam the stress-strain relation in compression is the same as that in tension. Subject this beam to a gradually increasing load until it breaks. Now the bending moment at the section where the beam fails may be computed since the loading is known. Call this bending moment the *ultimate bending moment* and designate it by M''. If the beam fails at the section BD (Fig. 262), then just before failure the stress

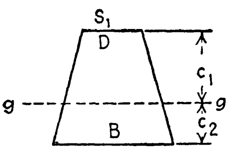


FIG. 260.

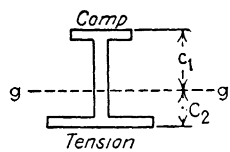


FIG. 261.

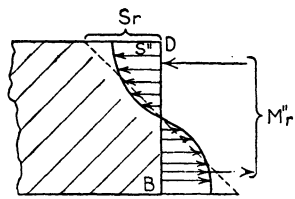


FIG. 262.

distribution is indicated by the heavy line. In the figure, S'' equals the ultimate stress in the outer fiber (actual stress at failure) and M_r'' equals the ultimate resisting moment.

There is no formula that will enable one to find S'', the ultimate fiber stress in flexure. The flexure formula does not apply since this formula assumes that the stress is triangularly distributed over the section as is indicated by the dotted line in the figure. It is convenient, therefore, to use a fictitious stress. This fictitious stress is called the *modulus of rupture* and is the stress that would exist in the outer fiber if at rupture (1) *the stress distribution were triangular,* and (2) *the neutral axis did not shift.* That is, by designating the modulus of rupture by S_r (see figure),

$$S_r = \frac{M''c}{I}$$

where M'' = bending moment that actually produced rupture in a test.

From the figure it is evident that $S_r > S''$. As stated above, we do not know the value of S''. It is customary, therefore,

to take the ultimate in direct compression or tension (the smaller of the two) as a basis for comparison. By designating the ultimate in direct compression or tension (no bending) by S_u, it has been found that S_r may be as much as 100 per cent larger than S_u.

The modulus of rupture S_r for a given material depends not only upon the properties of that material but also upon the shape of the section. Hence, to make the results of different tests comparable with each other, such tests are standardized. For instance, in the Forest Products Laboratory, the standard size for wood is taken as 2 by 2 by 28 in.

Ductile materials, such as soft steel, will bend without breaking. For such a material, the modulus of rupture has little meaning. The modulus of rupture is applied primarily to materials that will break when slightly bent. For a beam of such a material it may be used as a basis for selecting the safe fiber stress.

Illustration.—The ultimate strength S_u of Southern yellow pine in direct compression is (on the average) about 4000 lb./sq. in. The modulus of rupture S_r of this wood, if used as an ordinary beam, is about 8000 lb./sq. in. If now a factor of safety of 4 (say) is used for this wood in direct compression (no bending), thus making the allowable stress in direct compression

$$1000 \text{ lb./sq. in.},$$

it seems logical to use a factor of safety of 6 (say) applied to the modulus of rupture and thus make 1300 lb./sq. in. the allowable fiber stress in a beam of this material (see Art. 30, Table II, footnote).

PROBLEMS

139. A 14-in. 87-lb. WF beam has a plate, 10 by 1 in., riveted to its top flange. Locate the neutral axis N of the compound section. Also find I_N and the corresponding section modulus Z.

Ans. $\bar{y} = 9.11$ in. from bottom of beam; $I_N = 1372$ in.^{4}; $Z = 150.6$ in.^{3}

140. The beam of Problem 139 is 30 ft. long and rests on end supports. What uniformly distributed load can this beam carry if the allowable value of S is 18,000 lb./sq. in.? *Ans.* $W = 60{,}240$ lb.

141. Four planks, 3 in. wide, 2 in. thick, and 10 ft. long, are firmly nailed together to form a beam 3 in. wide and 8 in. high. If placed on end supports, what uniformly distributed load can this beam carry if $S = 1200$ lb./sq. in.? *Ans.* $W = 2560$ lb.

142. In Problem 141, if the boards had not been nailed together, what total load could the four boards carry? *Ans.* $W = 640$ lb.

143. In Problem 141, spikes 4 in. long are used to hold the boards together. Assuming that the safe lateral resistance of one spike is 200 lb., how many

spikes should be used near the support per foot of length of the beam to hold the two inner pieces together?

Ans. 15, or one spike to every 2.5 sq. in. of board surface.

Find a similar result for the quarter point of the span.

144. A beam is loaded as shown in Fig. 263. Neglect its weight. (*a*) Find a value of P so that the moment at B is numerically equal to the largest moment between O and B. (*b*) For this value of P draw the shear and moment diagrams. (*c*) Select an American Standard I-beam to carry the loads. $S = 18{,}000$ lb./sq. in.

Suggestion.—First find R_O in terms of P. Then locate the section of maximum moment along OB in terms of P.

Ans. $P = 2.28$ tons; a 12-in. 35-lb. I-beam.

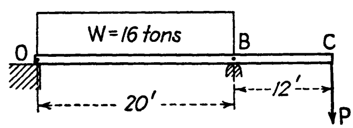


FIG. 263.

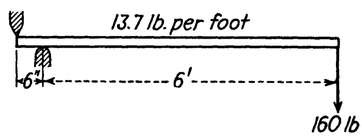


FIG. 264.

145. A $2\frac{1}{2}$-in. pipe is used as a lever (Fig. 264). The actual outside diameter of the pipe is 2.87 in. and the inside diameter is 1.77 in. The moment of inertia of the section of the pipe about a diameter is 2.85 in.^{4} A man weighing 160 lb. places his whole weight upon the outer end of the horizontal pipe. What is the maximum fiber stress in the pipe? Construct the shear and moment diagrams. *Ans.* 7,255 lb./sq. in.

146. Two loads, of 2 tons and of 3 tons, roll over a wooden beam 16 ft. long. The fixed distance between the two loads is 5 ft. For what position of the loads will the bending moment be a maximum? Find the maximum moment. The wooden beam is 6 in. wide and 16 in. high. What is the maximum fiber stress that will be induced in the beam?

Ans. 1435 lb./sq. in.

147. A 10-in. 49-lb. WF beam has the dimensions shown in Fig. 265. If V is the total shear acting on a section of this beam, find the unit shear (*a*) by means of the exact formula [Eq. (11), Art. 163]; (*b*) by means of the approximate formula [Eq. (14), Art. 165]. *Ans.* $0.331V$; $0.294V$.

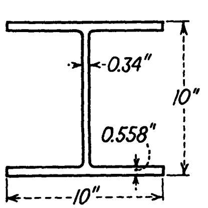


FIG. 265.

148. A wooden beam on end supports is 6 in. wide and 12 in. high. The beam carries a central load P. What is the greatest value P may have if the unit shear in the beam is not to exceed 125 lb./sq. in. [Eq. (12) Art. 164]? *Ans.* 12,000 lb.

149. A 15-in. 45-lb. American Standard I-beam, 16 ft. long and on end supports, was chosen to carry a central load of $P = 22{,}500$ lb. on the assumption that the load plane would be normal to the 1 . . . 1 axis. Afterward it was found that the load plane made an angle of $\theta = 3°$ with the 2 . . . 2 axis (see Fig. 223). What was the assumed maximum fiber stress and what was the actual fiber stress in the beam? *Ans.* 17,850 lb./sq. in.; 28,300 lb./sq. in.

150. In Problem 149, the load actually makes an angle of $\theta = 3°$ with the 2 . . . 2 axis. Locate the neutral axis N [Art. 159, Eq. (c)].

Ans. 57° 45′.

151. An area 8 in. square has a central circular hole 4 in. in diameter. (a) Compute the section modulus with respect to a gravity axis parallel to a side of the square. *Ans.* 82.2 in.³

(b) Compute a similar value with respect to a diagonal of the square.

Ans. 58.1 in.³

152. Compute the greatest value of the distance a at which the load can be placed if the stress in the wooden beam (Fig. 266), 6 in. wide and 8 in. high, is not to exceed 1200 lb./sq. in. Neglect the weight of the beam. *Ans.* a = 5.53 ft.

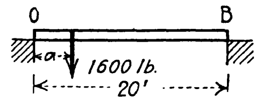


FIG. 266.

153. Take the weight of wood as 48 lb./cu. ft. What effect will the weight of the beam have upon the maximum fiber stress for the particular position of the load determined in Problem 152?

Ans. It increases maximum S by 120.8 lb./sq. in.

154. A floor beam of Norway pine is 2 in. wide and 10 in. high. The beam rests on end supports and is uniformly loaded. Take S = 1100 lb./sq. in. and S_s = 110 lb./sq. in. What is the span limit? That is, for what length will the beam be as strong in flexure as in shear? Do not use Eq. (c) of Art. 168. *Ans.* L = 8 ft. 04 in.

155. What is the maximum safe load the beam of Problem 154 can carry if 7 ft. long? If 9 ft. long? *Ans.* 2933 lb., 2716 lb.

156. A 30-in. 180-lb. WF beam of length L is to rest on end supports and is to be loaded to its full capacity. S = 18,000 lb./sq. in. and S_s = 12,000 lb./sq. in. What is the span limit? Given t = 0.67 in. and Z = 555.2 in.³

Ans. L = 13.8 ft.

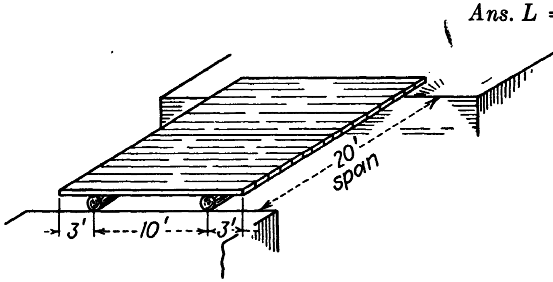


FIG. 267.

157. A temporary timber foot bridge consists of wooden planks laid on two logs as the sketch indicates. A loading uniformly distributed over the entire deck of the bridge and amounting to 90 lb./sq. ft. is to be carried safely. Design the planks and the logs for necessary bending strength. Use 800 lb./sq. in. bending stress. Consider the weight of the wood (40 lb./cu. ft.) when determining the size of the logs.

Ans. 2.5-in. planks; d = 18.8 in.

CHAPTER VIII

ELASTIC CURVE

DEFLECTION OF BEAMS

174. In the preceding chapter, simple beams were considered with special reference to the stresses induced in such beams when loaded. Now a beam subjected to a transverse load will bend more or less. That is, the transverse load will produce a "deflection."

In some cases a beam must be able to sustain loads without an undue deflection. In other cases, a certain amount of deflection is desired. For instance, floor beams must not sag to such an extent that the plastered ceiling of the room below will crack to an unsightly and unsafe degree. The limit for the deflection below which such cracking will not occur is usually specified as 1/360 of the span ($L/360$). On the other hand, a leaf of a spring used for a car is particularly dimensioned to give appreciable deflection. Many other instances in which the deflection of beams is important might be cited. It may be of interest to state here that the theory of deflection leads to the solution of statically indeterminate beams as will be seen at the end of this chapter.

The purpose of this chapter is to develop certain fundamental equations pertaining to deflection, and to apply these equations to a few of the simpler cases of beams. In Chap. IX, the moment area method for determining the deflection of beams will be considered.

The usual assumptions will be made in the analyses. The most important of these assumptions are here repeated for convenience of reference.

1. Originally the beam was straight, prismatic, and horizontal.
2. The deflection is relatively slight.
3. The elastic limit is not exceeded.

175. Elastic Curve.—The (straight) line going through the centroids of the sections of a beam in its unbent state is the *axis* of the beam. The curved line into which this axis changes

when the beam is bent is termed the *elastic curve* of the beam (Fig. 268). Equations for this curve are now to be developed.

Let CD and FG be two sections originally parallel and at a distance ds apart but now inclined (as an effect of the bending) so that their extensions meet at the point O as shown (Fig. 268). If the length ds of the elastic curve is considered as an arc of a circle, then O, the center of this circle, is called the *center of curvature* of the elastic curve at the point N (or N'). The

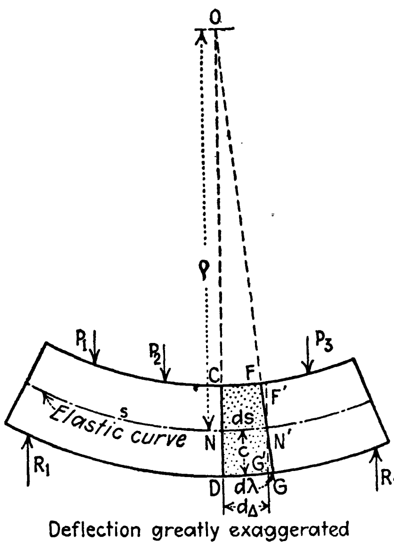


FIG. 268.

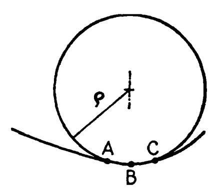


FIG. 268a.

radius of this circle is called the *radius of curvature* and will be designated by ρ.*

Through the point N' draw the line $F'G'$ parallel to CD. It is evident from the figure that the lower fiber has elongated an amount $d\lambda = G'G$. The relative or unit elongation of this fiber is

$$\epsilon = \frac{d\lambda}{ds}. \tag{a}$$

From similar triangles ($G'N'G$ and NON'),

* Draw a circle through three neighboring points in a curve (three points determine a circle). When the three points approach coincidence (say at B), then the circle is called the *osculating circle* at B, and the radius of this circle is the radius of curvature at the point B.

$$\frac{d\lambda}{ds} = \frac{c}{\rho},$$

Or [Eq. (*a*)]

$$\epsilon = \frac{c}{\rho}. \tag{b}$$

Now $E = S/\epsilon$ (Art. 26) and therefore

$$S = E\epsilon.$$

Or [Eq. (*b*)]

$$S = \frac{Ec}{\rho}.$$

Substituting this value of S in the flexure formula $M = SI/c$,

$$M = \frac{EI}{\rho}. \tag{1}$$

Note 1.—*If in a prismatic beam* (E *and* I *constants*) *the bending moment* M *is constant between two points, then the elastic curve between these two points takes the form of a circle* (ρ = constant).

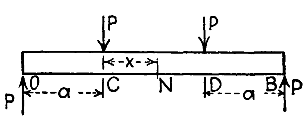


FIG. 269.

For instance, let Fig. 269 represent a beam symmetrically loaded as shown. The reactions are $R_O = P$ and $R_B = P$. The moment at the point N, a distance x from C, is $M = P(a + x) - Px = Pa$ = constant for all values of x between C and D. Hence the elastic curve of the beam between C and D is a circle of radius

$$\rho = \frac{EI}{Pa}.$$

Note 2.—In calculus it is shown that at a point of inflexion in a curve ρ is infinite.* Putting $\rho = \infty$ [Eq. (1)], $M = 0$. That is, the bending moment is zero at a point of inflexion in a beam. For instance, Fig. 270 represents an overhanging beam. At the point C in the elastic curve there is a point of inflexion. Hence the bending moment at C is zero. Note that the moment diagram crosses the base line under C. Conversely, since the moment diagram crosses the base line under C, C is a point of inflexion in the elastic curve.

Note 3.—If ρ is the radius of curvature at a point in a curve, then, by definition, $1/\rho$ equals *curvature at that point.* The

* At a point of inflection in a curve the concavity shifts from one side to the other side of the curve. Thus (Fig. 270), to the left of C, the concavity is above and to the right of C the concavity is below the beam.

curvature at a point in a curve is a measure of the bending of the curve at that point. If $\rho = \infty$, the curvature is zero; *i.e.*, there is no bending at that point in the curve. Hence, for a straight line, $\rho = \infty$.*

Example.—For a given beam, $I = 721$ in.^{4},

$$E = 30{,}000{,}000 \text{ lb./sq. in.},$$

and the maximum bending moment in the beam is

$$M = 180{,}000 \text{ ft.-lb.} = 2{,}160{,}000 \text{ in.-lb.}$$

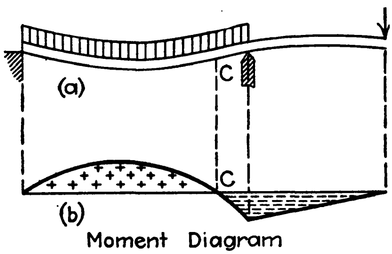


FIG. 270.

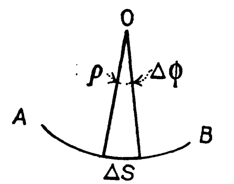


FIG. 271.

What is the radius of curvature at the section of maximum moment?

$$\rho = \frac{EI}{M} = 10{,}000 \text{ in.} = 833 \text{ ft.}$$

Hence the curvature is (using the foot as the unit),

$$\frac{1}{\rho} = \frac{1}{833} = 0.0012 \text{ (radians per foot of arc).}$$

176. Differential Equation of the Elastic Curve.—In calculus it is shown that, if ρ is the radius of curvature at a point (x, y) in a curve (Fig. 272),

* Let AB be the arc of a circle of radius ρ (Fig. 271). Let Δs be an increment of the arc and $\Delta\phi$ the angle subtended at the center of the circle. By definition,

$$\Delta\phi \text{ (in radians)} = \frac{\text{arc}}{\text{radius}} = \frac{\Delta s}{\rho}.$$

Or

$$\frac{\Delta\phi}{\Delta s} = \frac{1}{\rho}.$$

If $\Delta s = 1$, then $\Delta\phi = 1/\rho$; that is, $1/\rho$, the curvature of the circle, equals the angle subtended at the center of the circle per unit length of arc.

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{a}$$

in which dy/dx = slope of the tangent at the point $(x, y) = \tan \theta$.*

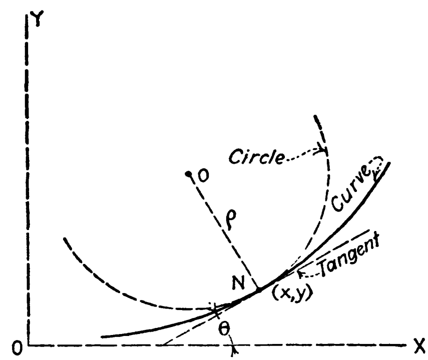


FIG. 272.

Equation (*a*) is applicable to any curve. When applied to the elastic curve of a beam (originally straight), this equation can be simplified. Take the X-axis parallel to the unbent

* At the point N (Fig. 272), $dy/dx = \tan \theta$. By differentiation

$$\frac{d^2y}{dx^2} = \sec^2 \theta \frac{d\theta}{dx} = (1 + \tan^2 \theta)\frac{d\theta}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{d\theta}{dx}.$$

Solving for $d\theta/dx$,

$$\frac{d\theta}{dx} = \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2} \tag{b}$$

From trigonometry,

$$\rho d\theta = ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Or

$$\frac{d\theta}{dx} = \frac{1}{\rho}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}. \tag{c}$$

Equating values of $d\theta/dx$ [Eqs. (*b*) and (*c*)] and solving for $1/\rho$

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

beam (Fig. 273). Since the deflection is small, the slope at any point in the elastic curve is so small that $(dy/dx)^2$ may be neglected when compared with unity. Hence, *if the X-axis is parallel to the unbent beam,* we may put

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}.$$

Substituting in Eq. (1), *i.e.*, in the equation

$$\frac{EI}{\rho} = M, \qquad (1)$$

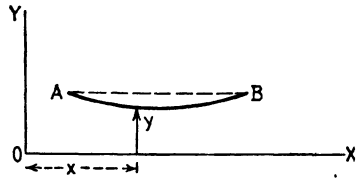


FIG. 273.

we obtain

$$EI\frac{d^2y}{dx^2} = M \qquad (2)$$

as the approximate differential equation of the elastic curve.

177. Rule of Sign.—When the equation

$$EI\frac{d^2y}{dx^2} = M$$

is used, the sign of M and that implied in d^2y/dx^2 must be considered.

It was agreed (Art. 120) to consider the bending moment M at a section in a beam as positive (+) if the concavity is above the beam (compression in the upper fibers) and negative (−) if below. *If the moment of an upward force is taken as positive and that of a downward force as negative, the sign of M will take care of itself.*

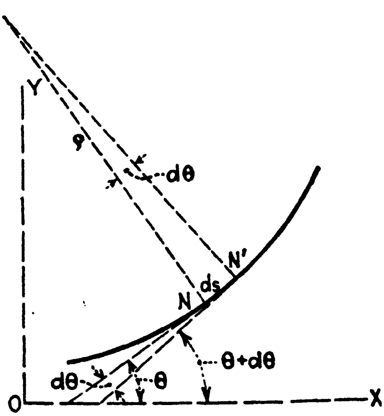


FIG. 274.

Figure 274 represents the elastic curve for a portion of a beam (bending much exaggerated, beam originally straight and horizontal). Since the concavity is above the beam, the bending moment at N is positive by assumption. Take O as the origin and draw the axes as shown. Note that the Y-axis is positive upward. As the point (x, y) moves from N to N', the angle θ, and therefore $\tan\theta$ or dy/dx, increases. If a

quantity increases as x increases, its derivative with respect to x is positive. Hence, if the concavity is above the beam and if y is measured upward as positive, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ is intrinsically positive; *i.e.*, M is positive by assumption, and d^2y/dx^2 is intrinsically positive. On the other hand, if the concavity is below the beam, M is negative by assumption, and d^2y/dx^2 is intrinsically negative. Hence, in applying Eq. (2) to a beam originally straight and horizontal, *sign will take care of itself if the following rule is observed:*

1. Take upward as the positive direction of y.
2. When determining the bending moment M at a section, take the moment of an upward force as positive and that of a downward force as negative.

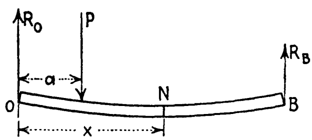


FIG. 275.

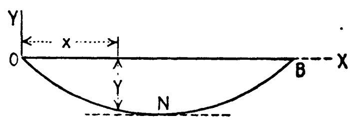


FIG. 276.

Illustration. Fig. 275.—At the point N

$$EI\frac{d^2y}{dx^2} = R_O x - P(x - a).$$

Note that, if y is positive upward, the rule of sign is observed and that therefore the equation is consistent as far as the implied signs are concerned.

178. Note on the Calculus.—The student is expected to be familiar with the calculus. The following discussion may be helpful. Let ONB (Fig. 276) be a curve whose equation is

$$y = ax^3 + bx^2 + C_1x + C_2 \qquad (a)$$

where a, b, C_1, and C_2 are known constants. If in this equation a value for x is substituted, the y of the corresponding point in the curve is determined.

Differentiating Eq. (a) with respect to x,

$$\frac{dy}{dx} = 3ax^2 + 2bx + C_1. \qquad (b)$$

Note that the constant C_2 disappears. Equation (b) gives the slope of the curve at the point (x, y) of the curve. Differ-

entiating Eq. (*b*),

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = 6ax + 2b. \tag{c}$$

Note that the constant C_1 disappears.

Assume now that Eq. (*c*) is given and that, starting with this equation, we are required to retrace our steps and find Eq. (*a*) This reverse process is called *integration.* Sometimes it is called *taking the anti-derivative.*

Integrating Eq. (*c*), taking the first anti-derivative, we obtain Eq. (*b*). Note that the constant C_1 must be introduced. This constant now is not known but must be determined from known conditions. For instance, if the values of x and dy/dx are known for a given point in the curve, say for the point N, these values may be substituted in Eq. (*b*) and the constant C_1 may be determined. With C_1 known, Eq. (*b*) is fully determined. Integrating Eq. (*b*), finding the second anti-derivative of Eq. (*c*), we obtain Eq. (*a*). Note that the constant C_2 is not known but must be determined from known conditions. For instance, if x and y are known for some point, say the point O, C_2 can be determined.

Note.—It may be of interest to note that in the integral

$$\int_b^y dy = \int_a^x f(x)dx$$

the lower limits merely state that $y = b$ when $x = a$, and that this integral is equivalent to saying that

$$y = \int f(x)dx + C$$

where C may be determined from the limiting condition that $y = b$ when $x = a$.

179. *Case* I. *Deflection of Simple Beam on End Supports.* Fig. 277.—Central load P. E and I constant. Weight of beam neglected.

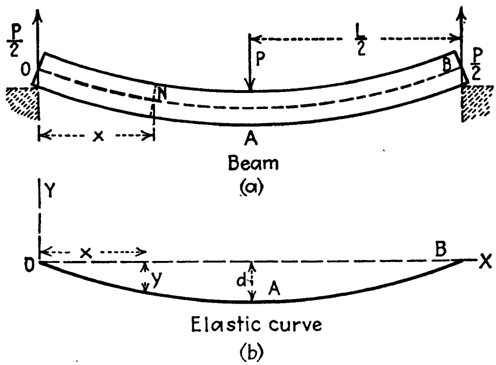
Each of the pier reactions is $P/2$. Take O as the origin of coordinates and y as positive upward. Let N be any point between O and A, A being the center of the beam. Considering forces to the left of the section, the bending moment at N is $M_x = Px/2$. Since the rule of sign is observed (Art. 177),

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}x.$$

Integrating

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1 \qquad (a)$$

in which C_1 is a constant of integration. Equation (a) holds for any point between O and A and therefore it holds for the point A. At A the tangent to the elastic curve is horizontal



FIGS. 277*a* and *b*.

(zero slope). Putting $x = L/2$ and $dy/dx = 0$ in Eq. (a)

$$EI(0) = \frac{PL^2}{16} + C_1 \qquad \text{or} \qquad C_1 = -\frac{PL^2}{16}.$$

Substituting this value of C_1 in Eq. (a)

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{PL^2}{16}. \qquad (b)$$

Integrating again,

$$EIy = \frac{Px^3}{12} - \frac{PL^2x}{16} + C_2. \qquad (c)$$

This equation holds for any point between O and A. It holds therefore at the point O where both x and y are zero. Putting $x = 0$ and $y = 0$ in Eq. (c), $C_2 = 0$. Therefore

$$EIy = \frac{Px^3}{12} - \frac{PL^2x}{16}. \qquad (d)$$

This is the equation of the elastic curve between O and A with reference to an origin of coordinates at O.*

* The equation of the elastic curve between A and B with O as the origin of coordinates takes a different form since, for a point between A and B,

$$M = \frac{P}{2}x - P\left(x - \frac{L}{2}\right)$$

Now, $y = 0$ when $x = L$.

The maximum deflection is at A, the center of the beam. Hence, putting $x = L/2$ in Eq. (d).

$$y_{\text{max.}} = \frac{1}{EI}\left(\frac{PL^3}{96} - \frac{PL^3}{32}\right) = -\frac{1}{48}\frac{PL^3}{EI}.$$

The minus sign indicates that the deflection of A is opposite to the positive direction of y. If this is understood, the minus sign may be omitted. With the maximum value of y represented numerically by d, we have

$$d = \frac{PL^3}{48EI}. \tag{3}$$

Note.—When Eq. (3) is used, it is necessary to be consistent in the use of units. If E is in pounds per square inch, and I in

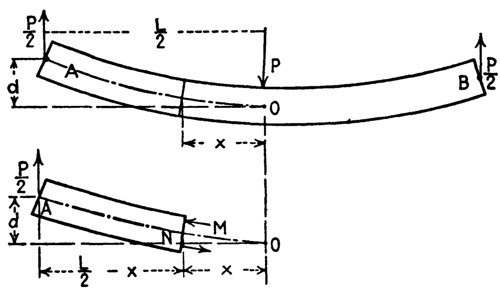


FIG. 277c.

inches[4], P must be in pounds and L in inches. The deflection d will then be expressed in inches.

Problem 158.—If the center of the beam in its deflected position is taken as the origin (Fig. 277c)

$$M_x = \frac{P}{2}\left(\frac{L}{2} - x\right).$$

Derive the equation of the elastic curve.

$$EIy = \frac{PLx^2}{8} - \frac{Px^3}{12}.$$

Show that

$$y_{\text{max.}} = +\frac{1}{48}\frac{PL^3}{EI}.$$

180. *Case* II. *Simple Beam on End Supports. Load Uniformly Distributed.* Fig. 278.—Taking O as the origin, the bending moment at the section N (any point in the beam) is

$$M_x = \frac{wLx}{2} - \frac{wx^2}{2}.$$

Hence

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}.$$

Integrating,

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1. \qquad (a)$$

At the point A, the center of the beam, $dy/dx = 0$ and $x = L/2$. Substituting these values in Eq. (a) and solving for C_1,

$$C_1 = -\frac{wL^3}{16} + \frac{wL^3}{48} = -\frac{wL^3}{24}.$$

Therefore

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24}. \qquad (b)$$

Integrating again,

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} + C_2. \qquad (c)$$

At the point O, $x = 0$ and $y = 0$. Hence $C_2 = 0$.
Finally,

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}. \qquad (d)$$

Note.—Since N may be any point in the beam, the point B may be taken to determine C_2. At B, $y = 0$ and $x = L$. Substituting in Eq. (c), $C_2 = 0$.

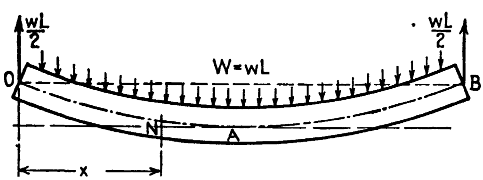


FIG. 278.

The maximum deflection is at A, the center of the beam. Putting $x = L/2$ in Eq. (d)

$$y_{max.} = \frac{1}{EI}\left(\frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48}\right) = -\frac{5wL^4}{384EI} = -\frac{5}{384}\frac{WL^3}{EI}.$$

Therefore, if d designates the numerical value of the deflection,

$$d = \frac{5}{384}\frac{wL^4}{EI} = \frac{5}{384}\frac{WL^3}{EI}. \qquad (4)$$

Problem 159.—If the center of the beam is taken as the origin, show that

$$M = \frac{wL}{2}\left(\frac{L}{2} - x\right) - \frac{w}{2}\left(\frac{L}{2} - x\right)^2,$$

and that the equation of the elastic curve becomes

$$EIy = \frac{w}{2}\left(\frac{L^2x^2}{8} - \frac{x^4}{12}\right).$$

181. *Case* III. *Cantilever Beam with Concentrated Load P at End.* Fig. 279.—Take the origin O at the free end of the beam in its position before the load is applied. At the section N, distant x

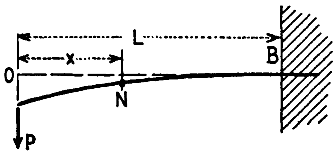


Fig. 279.

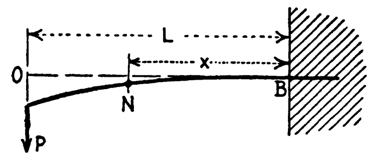


Fig. 280.

from O, the moment is $M_x = -Px$ (the moment of a downward force is negative).

$$EI\frac{d^2y}{dx^2} = -Px;$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1.$$

At B, $x = L$ and $dy/dx = 0$. Therefore

$$C_1 = \frac{PL^2}{2};$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + \frac{PL^2}{2};$$

$$EIy = -\frac{Px^3}{6} + \frac{PL^2x}{2} + C_2.$$

At B, $x = L$ and $y = 0$. Therefore

$$C_2 = -\frac{PL^3}{3}.$$

Finally,

$$EIy = -\frac{Px^3}{6} + \frac{PL^2x}{2} - \frac{PL^3}{3}. \qquad (a)$$

This is the equation of the elastic curve. Therefore

$$y_{\max} = -\frac{PL^3}{3EI}. \qquad d = \frac{PL^3}{3EI}. \qquad (5)$$

Problem 160.—Taking B as the origin (Fig. 280), $M_x = -P(L - x)$, and the equation of the elastic curve is

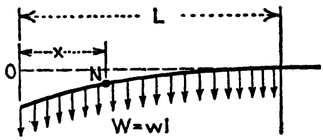


Fig. 281.

$$EIy = -\frac{PLx^2}{2} + \frac{Px^3}{6}. \qquad \text{(Show.)}$$

182. *Case* IV. *Cantilever Beam with Uniformly Distributed Load.* Fig. 281.—

$$M_x = -\frac{wx^2}{2} = EI\frac{d^2y}{dx^2};$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wL^3}{6};$$

$$EIy = -\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}. \qquad (a)$$

$$y_{\text{max.}} = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI}$$

Therefore

$$d = \frac{wL^4}{8EI} = \frac{WL^3}{8EI}. \qquad (6)$$

183. *Case* V. *Simple Beam on End Supports Bearing a Single Concentrated Load P in an Eccentric Position.* Fig. 282.—To find

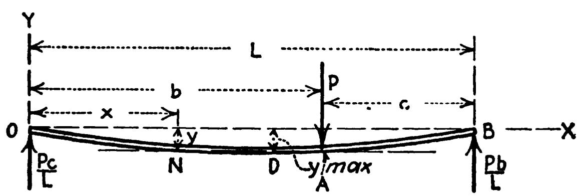


Fig. 282.

R_O, the reaction at O, consider the whole beam free and put Σ moments about B equal to zero. Similarly, Σ moments about O gives R_B.

$$R_O = \frac{Pc}{L} \quad \text{and} \quad R_B = \frac{Pb}{L}.$$

The bending moment at a section N that lies to the left of the load is $M_x = Pcx/L$ in which x may have any value between 0 and b. The moment at a section that lies to the right of P is

$$M_x = \frac{Pcx}{L} - P(x - b)$$

in which x may have any value between b and L. Accordingly,

To the left of the load

$$EI\frac{d^2y}{dx^2} = \frac{Pcx}{L}. \qquad (a)$$

$$EI\frac{dy}{dx} = \frac{Pcx^2}{2L} + C_1. \qquad (b)$$

$$EIy = \frac{Pcx^3}{6L} + C_1x + C_2. \qquad (c)$$

To the right of the load

$$EI\frac{d^2y}{dx^2} = \frac{Pcx}{L} - P(x - b). \qquad (a')$$

$$EI\frac{dy}{dx} = \frac{Pcx^2}{2L} - \frac{P(x - b)^2}{2} + C_1'. \qquad (b')$$

$$EIy = \frac{Pcx^3}{6L} - \frac{P(x - b)^3}{6} + C_1'x + C_2'. \qquad (c')$$

Note particularly the integration of the term in $(x - b)$.*

Equation (c) is the equation of the elastic curve OA and equation (c') is the equation of the curve AB. Each of these equations contains two undetermined constants which must be found. To determine the four constants C_1, C_1', C_2, and C_2', we shall make use of four known conditions; *viz.*,

1. At the point A there is a common tangent so that dy/dx of curve OA equals dy/dx of curve AB.
2. At O, the deflection is zero; $y = 0$ when $x = 0$.
3. At A, the y of curve OA equals the y of curve AB.
4. At B, the deflection is zero; $y = 0$ where $x = L$.

First Condition.—If x is made equal to b in Eqs. (b) and (b'), the right-hand members of these equations are equal.

$$\frac{Pcb^2}{2L} + C_1 = \frac{Pcb^2}{2L} - 0 + C_1'.$$

* The integral $\int(ax + b)^n dx$ is evaluated in a simple manner by making use of the formula

$$\int u^n du = \frac{u^{n+1}}{n + 1}.$$

Note that, if $u = ax + b$, $du = adx$. To evaluate the given integral, proceed, therefore, as follows:

$$\int(ax + b)^n dx = \frac{1}{a}\int(ax + b)^n adx = \frac{1}{a}\int(ax + b)^n d(ax + b) = \frac{1}{a}\frac{(ax + b)^{n+1}}{n + 1}.$$

For instance,

$$\int(x - b)^2 dx = \int(x - b)^2 d(x - b) = \frac{(x - b)^3}{3}.$$

Therefore

$$C_1 = C_1'.$$

Second Condition.—If in Eq. (c) x is put equal to zero, y also is zero. Therefore

$$C_2 = 0.$$

Third Condition.—If x is made equal to b, the right-hand member of Eq. (c) is equal to the right-hand member of Eq. (c'). That is, since $C_1' = C_1$ and $C_2 = 0$ as proved above,

$$\frac{Pcb^3}{6L} + C_1 b = \frac{Pcb^3}{6L} - 0 + C_1 b + C_2'.$$

Therefore

$$C_2' = 0.$$

Accordingly Eqs. (c) and (c') may be written ($C_1 = C_1'$, $C_2 = 0$, $C_2' = 0$)

$$EIy = \frac{Pcx^3}{6L} + C_1 x. \tag{d}$$

$$EIy = \frac{Pcx^3}{6L} - \frac{P(x - b)^3}{6} + C_1 x \tag{d'}$$

Fourth Condition.—If x has the value L, y in Eq. (d') equals zero.

$$0 = \frac{PcL^3}{6L} - \frac{P(L - b)^3}{6} + C_1 L.$$

Or

$$C_1 = -\frac{PcL^2}{6L} + \frac{P(L - b)^3}{6L}.$$

Since $L - b = c$, this may be simplified.

$$C_1 = -\frac{PcL^2}{6L} + \frac{Pc^3}{6L} = -\frac{Pc}{6L}(L^2 - c^2).$$

If this value of C_1 is substituted in Eq. (d), we have

$$EIy = \frac{Pcx^3}{6L} - \frac{Pc(L^2 - c^2)x}{6L}. \tag{e}$$

This is the equation of the elastic curve for the part OA.

In (Fig. 282), b is larger than c; hence $y_{max.}$ occurs to the left of P. If D is the point at which the maximum ordinate is found (see figure) then at D the slope dy/dx is zero. Differentiating Eq. (e) and putting $dy/dx = 0$,

$$EI0 = \frac{Pcx^2}{2L} - \frac{Pc(L^2 - c^2)}{6L}.$$

Solving for x,

$$x_1 = \sqrt{\frac{L^2 - c^2}{3}}. \tag{f}$$

This locates the point of maximum deflection x_1 measured from O. If this value of x is substituted in Eq. (*e*), $y_{max.}$ is determined and is found to be

$$y_{max.} = \frac{Pc(L^2 - c^2)^{\frac{3}{2}}}{9EIL\sqrt{3}}. \tag{g}$$

If c is larger than b, take point B as the origin. The result is to replace c by b in Eqs. (*f*) and (*g*) and to measure x from B.

From Fig. 282 it is seen that there are two limiting cases; *viz.*, $c = 0$ and $c = L/2$. If in Eq. (*f*), $c = 0$, then $x_1 = L/\sqrt{3}$; and if $c = L/2$, $x_1 = L/2$. It thus appears that the point in the elastic curve that has the greatest deflection cannot be more than $L/\sqrt{3} - L/2$ from the middle of the span, *i.e.*, $0.0733L$ or nearly $L/13$. For instance, if the beam is 13 ft. in length, the point of maximum deflection lies within 1 ft. from the middle of the beam. Moreover, at the point of maximum deflection the tangent to the elastic curve is horizontal and in the neighborhood of that point y does not vary appreciably.

In general, *no appreciable error is made by assuming that the maximum deflections of a simple beam on end support occurs at the middle of the span for any system of loading.*

CONTINUOUS BEAMS

184. The beams so far considered are statically determinate; *i.e.*, the available equations of equilibrium are sufficient in number to determine the reactions. Thus for simple beams there are two reactions. These may be determined from the two equations $\Sigma M_0 = 0$ and $\Sigma M_B = 0$.

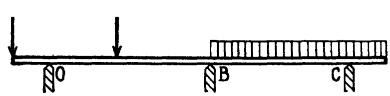


FIG. 283.

Frequently a beam is supported at three or more points (Fig. 283) or is "built in" at one end and simply supported at one or more points (Fig. 284*a*) or is built in at both ends (Fig. 285*a*).

Such beams are *statically indeterminate.* There are at least three unknowns (reactions) and at most two equations of equili-

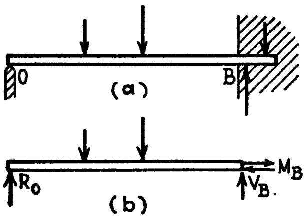


FIG. 284.

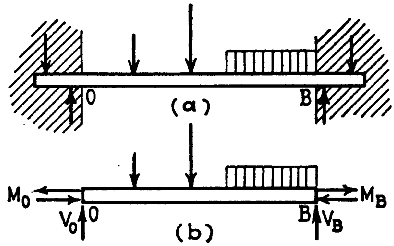


FIG. 285.

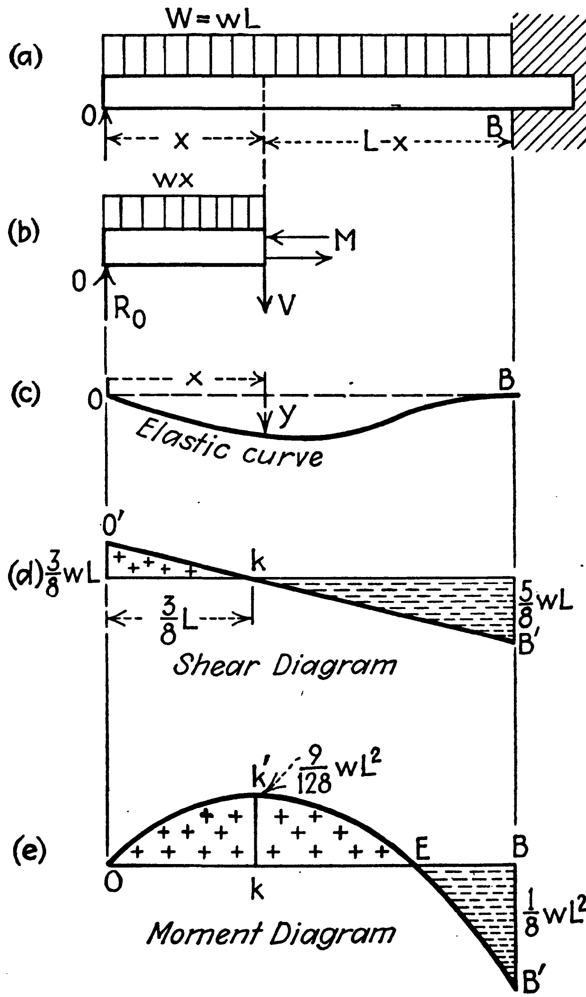


FIG. 286.

brium to determine them (Art. 111). Statically indeterminate beams often are called *continuous beams.*

In dealing with a continuous beam with one end built in (Fig. 284*a*), it is simpler to cut the beam just to the left of B and to take the part of the beam to the left of B free as is shown in Fig. 284*b*, where M_B is the moment of the couple and V_B is the shear acting on the section at B. For convenience the three unknowns R_O, V_B, and M_B will be called the three *reactions.* In like manner, Fig. 285*a* represents a continuous beam with both ends built in, and Fig. 285*b* represents this beam taken free. The four unknowns V_O, M_O, V_B, and M_B will be called the four *reactions.*

To determine the reactions of a continuous beam it is necessary to consider the deflection. This is equivalent to saying that the reactions of a continuous beam depend upon the way the beam bends.

In the remaining articles of this chapter, a few of the simpler cases of continuous beams will be considered primarily for the purpose of showing how the equation of the elastic curve may be used to determine the reactions. In Chap. IX, a simpler method of procedure will be used and continuous beams loaded in various ways will there be considered.

185. Fixed End.—If a built-in beam is so firmly held that the tangent to the elastic curve at the built-in end does not change its direction when the beam is loaded, the end is said to be *fixed.* In this chapter, unless a statement is made to the contrary, *built-in ends* are assumed to be *fixed horizontally.* That is, at a built-in end, $dy/dx = 0$.

186. *Case* I. *Beam with One End Fixed Horizontally. Other End Resting on Support. Uniform Load. O and B at the Same Level.* Fig. 286.—If the axes are chosen as indicated in the figure, O being the origin,

$$M_x = R_O x - \frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = R_O x - \frac{wx^2}{2}. \tag{a}$$

Figure *c* gives the elastic curve. Note that there are three limiting conditions,

(1) $dy/dx = 0$ when $x = L$.
(2) $y = 0$ when $x = 0$.
(3) $y = 0$ when $x = L$.

These three limiting conditions suffice to determine the two constants of integration C_1 and C_2 and the reaction R_O.

Integrating Eq. (a),

$$EI\frac{dy}{dx} = \frac{R_O x^2}{2} - \frac{wx^3}{6} + C_1. \tag{b}$$

At B the tangent to the elastic curve is horizontal. Putting $dy/dx = 0$ and $x = L$ (first limiting condition),

$$0 = \frac{R_O L^2}{2} - \frac{wL^3}{6} + C_1.$$

Therefore

$$C_1 = -\frac{R_O L^2}{2} + \frac{wL^3}{6}.$$

Substituting this value of C_1 into Eq. (b),

$$EI\frac{dy}{dx} = \frac{R_O x^2}{2} - \frac{wx^3}{6} - \frac{R_O L^2}{2} + \frac{wL^3}{6}.$$

Integrating again,

$$EIy = \frac{R_O x^3}{6} - \frac{wx^4}{24} - \frac{R_O L^2 x}{2} + \frac{wL^3 x}{6} + (C_2 = 0). \tag{c}$$

At O, $x = 0$, and $y = 0$ (second limiting condition). Therefore $C_2 = 0$.

To determine R_O, use the third condition, $y = 0$ where $x = L$.

$$0 = \frac{R_O L^3}{6} - \frac{wL^4}{24} - \frac{R_O L^3}{2} + \frac{wL^4}{6}.$$

Or, solving for R_O,

$$R_O = \frac{3}{8}wL = \frac{3}{8}W.^* \tag{d}$$

187. *Case* I (*Continued*). *Shear and Moment Diagrams, Etc.* Fig. 286.—Knowing R_O, we can now proceed exactly as was done for simple beams (Art. 130).

* Assume that the support O sinks an amount d and that the built-in end rotates counterclockwise through an angle α (in radians) as shown in Fig. 287. With O as the origin and upward as positive, the limiting conditions are

(1) $dy/dx = \alpha$ when $x = L$.
(2) $y = 0$ when $x = 0$.
(3) $y = d$ when $x = L$.

The three limiting conditions suffice to determine the two constants of integration C_1 and C_2 and the reaction R_O.

1. *Reaction* at O is $R_O = \frac{3}{8}wL = \frac{3}{8}W$.

2. *Shear Diagram.*—The vertical shear just to the right of O is $V = \frac{3}{8}wL$; just to the left of B, $V = \frac{3}{8}wL - wL = -\frac{5}{8}wL$. Since the shear diagram under a uniformly distributed load is a sloping straight line, draw the straight line $O'B'$ (Fig. 286*d*).

The shear is a maximum (numerically the largest) at B. Therefore

$$V_m = \frac{5}{8}wL. \tag{a}$$

3. *Maximum Bending Moment.*—The moment is a maximum (numerically the largest) either at k or at B. To locate k, put

$$V_k = \tfrac{3}{8}wL - wx = 0.$$

That is, k is $\frac{3}{8}L$ from O.

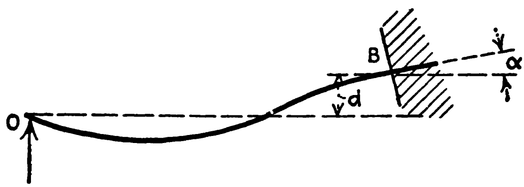


Fig. 287.

Therefore

$$M_k{}^* = \left(\frac{3}{8}wL\right) \times \left(\frac{3}{8}L\right) - \frac{w\left(\frac{3}{8}L\right)^2}{2} = \frac{9}{128}wL^2.$$

The moment at B is

$$M_B = \left(\frac{3}{8}wL\right)L - \frac{wL^2}{2} = -\frac{1}{8}wL^2 = -\frac{1}{8}WL. \tag{b}$$

The moment is a maximum at B.

$$M_m = \tfrac{1}{8}WL. \tag{c}$$

4. *Moment Diagram.*—Plot the moments M_k and M_B (Fig. 286*e* and draw the curve $Ok'B'$ to represent a parabola. (The moment under a uniformly distributed load is a parabola.)

* The shear area theorem (Art. 125) may be used to advantage here.

M_k = area of shear diagram between O and $k = \frac{1}{2} \times \left(\frac{3}{8}wL \times \frac{3}{8}L\right) = \frac{9}{128}wL^2$.

As a rule, the shape of the parabola may be approximated. Note, however, that the tangent to the curve at k' should be horizontal. At E the moment goes through zero. Hence the point E in the elastic curve is a point of inflection. That is, at E the concavity shifts from one side of the beam to the other side.

Problem 161.—Locate the point of inflection for the beam of Fig. 286.
Ans. $x = \frac{3}{4}L$.

188. *Case* II. *Beam on Three Supports at the Same Level. Spans of Equal Length. Load Uniformly Distributed.* Fig. 288.—

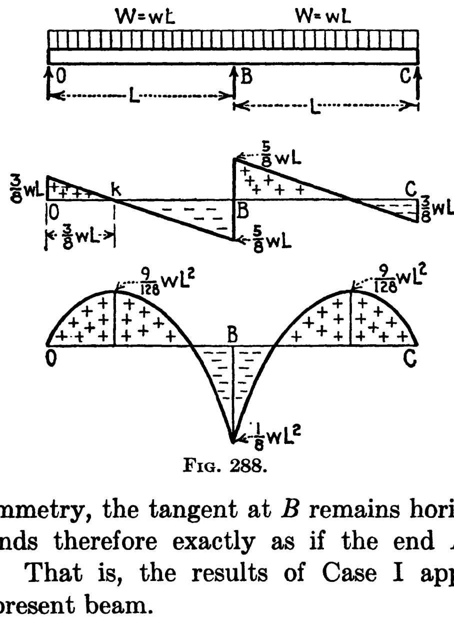


FIG. 288.

Owing to symmetry, the tangent at B remains horizontal. The span OB bends therefore exactly as if the end B were fixed horizontally. That is, the results of Case I apply to either span of the present beam.
Therefore

$$R_O = \frac{3}{8}wL = \frac{3}{8}W = R_C. \tag{a}$$

From $\Sigma F_y = 0$,

$$R_O + R_B + R_C - 2wL = 0.$$

Or

$$R_B = \tfrac{10}{8}wL = \tfrac{10}{8}W. \tag{b}$$

The moment is (numerically) a maximum at B.

$$M_m = \tfrac{1}{8}wL^2 = \tfrac{1}{8}WL. \tag{c}$$

189. *Case* III. *Beam Fixed at Both Ends. Supports at the Same Level. Load Uniformly Distributed.* Fig. 289.—Take OB free (Fig. 289*b*) and consider V_O, M_O, V_B, and M_B as four reactions

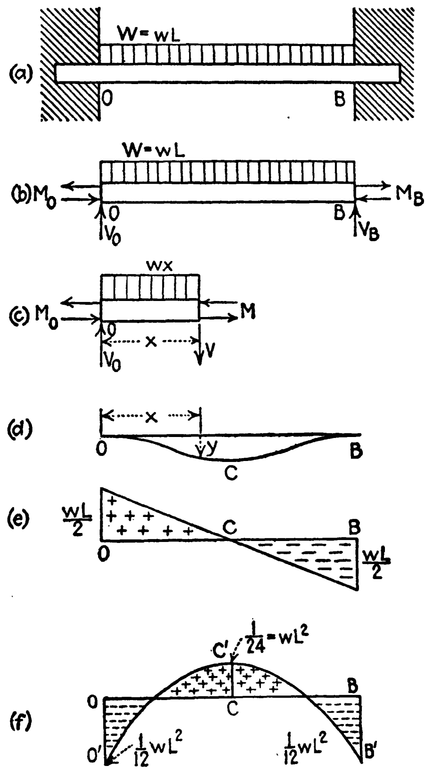


FIG. 289.

to be determined. With O as the origin, the bending moment at a section distant x from O is (Fig. 289*c*)

$$M_x = V_O x - M_O - \frac{wx^2}{2}. \tag{a}$$

Therefore

$$EI\frac{d^2y}{dx^2} = V_O x - M_O - \frac{wx^2}{2}. \tag{b}$$

This equation contains the two unknowns V_O and M_O; and the integration introduces the two constants of integration C_1 and C_2. We need then four limiting conditions.

In this particular case, V_O may be determined directly from the principle of symmetry. That is, from symmetry,

$$V_O = V_B = \frac{W}{2} = \frac{wL}{2}.$$

Substituting this value of V_O in Eq. (*b*),

$$EI\frac{d^2y}{dx^2} = \frac{wL}{2}x - M_O - \frac{wx^2}{2}.$$

Only three limiting conditions are now required. Integrating,

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - M_Ox - \frac{wx^3}{6} + (C_1 = 0). \qquad (c)$$

At O, $x = 0$ and $dy/dx = 0$ (see Fig. 289*d*).
Therefore

$$C_1 = 0.$$

To determine M_O, use the condition that at B the tangent to the elastic curve is horizontal. That is, in Eq. (*c*) put $dy/dx = 0$, and $x = L$.

$$0 = \frac{wL^3}{4} - M_OL - \frac{wL^3}{6}.$$

Or

$$M_O = \tfrac{1}{12}wL^2 = \tfrac{1}{12}WL. \qquad (d)$$

Substituting this value of M_O in Eq. (*c*),

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wL^2x}{12} - \frac{wx^3}{6}.$$

Integrating,

$$EIy = \frac{wLx^3}{12} - \frac{wL^2x^2}{24} - \frac{wx^4}{24} + (C_2 = 0). \qquad (e)$$

At O, $x = 0$, $y = 0$. Therefore $C_2 = 0$.
The maximum deflection occurs at the middle of the span. Putting $x = L/2$ in Eq. (*e*),

$$EIy_{\text{max.}} = \frac{wL^4}{96} - \frac{wL^4}{96} - \frac{wL^4}{384} = -\frac{wL^4}{384}.$$

Therefore

$$d = \frac{wL^4}{384EI} = \frac{WL^3}{384EI}. \qquad (f)$$

190. Note.—Referring to Fig. 289*b*, note that M_O, V_O, M_B, and V_B are considered external reactions. When the beam is taken free, these external reactions must be represented. As a rule, it is found convenient to represent a reaction as acting

in its true direction. If this direction is not apparent, it must be assumed.

In general, if the equations of equilibrium are written for an assumed direction of a reaction and if the sign of this reaction as resulting from the solution of equations of equilibrium (or of equations derived from them) is *positive*, the assumed direction of the reaction is correct. If the sign is negative, the reaction acts in the opposite direction. For instance, the couple at O (Fig. 289*b*) is drawn so as to produce tension in the upper fibers. On this basis the moment at O, *i.e.*, M_O, is found to be equal to $+\frac{1}{12}WL$ [Eq. (*d*)], the plus sign indicating that the direction of the couple is correct.

It was agreed to designate a moment as negative if it produces tension in the upper fibers (Art. 120). Hence, after both the direction and the magnitude of the moment at O have been determined, this moment may be written

$$M_O = -\tfrac{1}{12}WL, \tag{a}$$

the minus sign *now* indicating that the moment produces tension in the upper fibers.

It is very important to distinguish between the sign of a force or a moment resulting from the solution of an equation and the sign that *later* must be given this force or this moment so as to be consistent with the convention of signs. *When the moment diagram is constructed, the convention of sign must be followed.* Hence in the moment diagram M_O must be represented as a negative moment.

191. *Case* III (*Continued*). *Shear and Moment Diagrams, Etc.* Fig. 289.

(1) $$V_O = \frac{W}{2}. \qquad M_O = -\frac{1}{12}WL.$$

(2) *Shear Diagram.* Fig. 289*e*.—The shear diagram under a uniformly distributed load is a sloping straight line.

$$V_m = \frac{W}{2}.$$

(3) *Maximum Bending Moment.*—The moment at the middle of the beam is

$$M_C = \frac{W}{2} \times \frac{L}{2} - \frac{1}{12}WL - \frac{W}{2} \times \frac{L}{4} = +\frac{1}{24}WL.$$

Hence the moment (numerically) is a maximum at the supports and is (without regard to sign)

$$M_m = \tfrac{1}{12}WL.$$

(4) *Moment Diagram.*—The bending moment at O is $-\tfrac{1}{12}WL$, and at C it is $+\tfrac{1}{24}WL$. Hence (Fig. 289*f*) draw $O'C'B'$ to represent a parabola. (The moment under a uniformly distributed load is a parabola.)

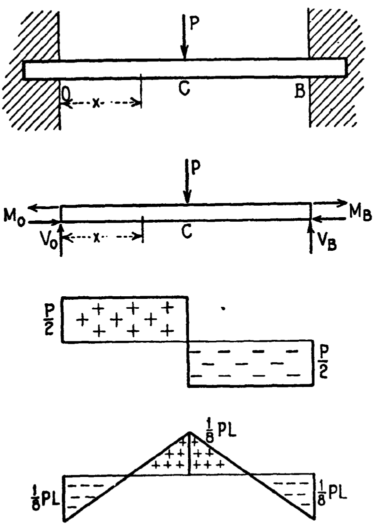


FIG. 290.

192. *Case* IV. *Beam Fixed Horizontally at Both Ends. Supports at the Same Level. Concentrated Load at Middle of Span.* Fig. 290.—The student should verify the following results:

$$V_O = \frac{P}{2}, \qquad V_B = \frac{P}{2},$$

$$M_x = \frac{P}{2}x - M_O.$$

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}x - M_O.$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - M_Ox + (C_1 = 0).$$

$$M_O = \frac{PL}{8}.$$

The plus sign indicates that the couple at O is properly drawn.

$$EIy = \frac{Px^3}{12} - \frac{PL}{8} \times \frac{x^2}{2} + (C_2 = 0).$$

$$d = \frac{1}{192}\frac{PL^3}{EI} \text{ (numerically).} \qquad (a)$$

$$M_C = \frac{P}{2} \times \frac{L}{2} - \frac{PL}{8} = \frac{PL}{8};$$

$$M_m = \frac{PL}{8}. \qquad (b)$$

If the convention of sign is followed,

$$M_O = -\frac{PL}{8} \quad \text{and} \quad M_C = +\frac{PL}{8}.$$

193. Advantages of Continuous Beams.—The equation of safe loading is

$$\text{Max. } M = M_m = \frac{S'I}{c}. \qquad (a)$$

Given a horizontal beam of length L. If the ends of this beam are built in horizontally (vertical load P_1 concentrated at the middle, Art. 192),

$$M_m = \frac{P_1 L}{8}.$$

Therefore [Eq. (*a*)]

$$\frac{P_1 L}{8} = S'\frac{I}{c}.$$

Or

$$P_1 = \frac{8S'I}{Lc}. \qquad (b)$$

Assume now that the ends of this beam rest on supports (P_2 concentrated at the middle, Art. 142, Fig. 284),

$$M_m = \frac{P_2 L}{4}.$$

Or [Eq. (*a*)]

$$\frac{P_2 L}{4} = \frac{S'I}{c}.$$

Or

$$P_2 = \frac{4S'I}{Lc}. \qquad (c)$$

By comparing Eqs. (*b*) and (*c*), it is seen that $P_1 = 2P_2$. That is, the beam with ends built in can carry twice the load that the same beam can carry with ends resting on supports (loads concentrated at the middle).

On the other hand, the deflection of the first beam is (Art. 192)

$$d_1 = \frac{1}{192}\frac{P_1 L^3}{EI},$$

while that of the second beam is (Art. 179)

$$d_2 = \frac{1}{48}\frac{P_2 L^3}{EI}.$$

Hence, since $P_1 = 2P_2$, $d_1 = d_2/2$. That is, the deflection of the beam when built in is one-half that of this beam when resting on end supports, the built-in beam carrying twice the load that the same beam can carry when resting on end supports.

In like manner, it can be shown that, if the beam is uniformly loaded,

$$W_1 = 1.5W_2 \qquad \text{and} \qquad d_1 = 0.30d_2.$$

Note.—The tendency in modern steel construction is to use continuous beams with the ends riveted or welded to the vertical frame, thus making the whole structure statically indeterminate. This increases the rigidity of the structure very much and results in increased strength or in economy of material.

194. Method of Equating Deflections.—An examination of the equations of the elastic curves developed in the foregoing articles will show that the deflection of a *given* point in a beam is directly proportional to the load producing this deflection. For instance (Art. 179), if a simple beam on end supports is loaded centrally, the equation of the elastic curve is

$$y = \frac{P}{EI}\left(\frac{x^3}{12} - \frac{L^2x}{16}\right).$$

Hence, for a specified value of x, y is directly proportional to P. If P is doubled, y is doubled. The principle of superposition may be used, therefore, to determine the deflection due to two or more loads (Art. 52). That is, the deflection of a point in a beam due to two or more loads equals the algebraic sum of the deflections of this point if each load in turn acts alone. For instance, if d_1 equals the deflection of a point if a load P_1 acts alone, and d_2 equals the deflection of this point if P_2 acts alone, then $d = d_1 + d_2$, equals the deflection of this point if both loads act simultaneously, provided the elastic limit is not exceeded. Frequently, this principle may be used to determine the reactions for a statically indeterminate beam as the following examples will illustrate.

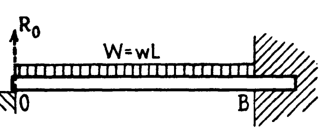


FIG. 291.

Example I. Fig. 291.—Beam with one end fixed horizontally and the other end resting on a support. Supports at the same level. Load uniformly distributed. If the support O is removed, then the downward deflection of O is

$$d_1 = \frac{1}{8}\frac{WL^3}{EI} \text{ (Art. 182).}$$

If the load W is removed and R_O is assumed to act, then the upward deflection of O would be

$$d_2 = \frac{1}{3}\frac{R_O L^3}{EI} \text{ (Art. 181).}$$

Since the point O does not move, the two deflections are equal. Hence

$$\frac{1}{8}\frac{WL^3}{EI} = \frac{1}{3}\frac{R_O L^3}{EI}.$$

Therefore

$$R_O = \tfrac{3}{8}W.$$

This agrees with the result obtained in Art. 186.

Example II. Fig. 292.—Beam on three supports. Supports at the same level. Spans equal. Load uniformly distributed.

If the support B is removed, the downward deflection of the point B would be

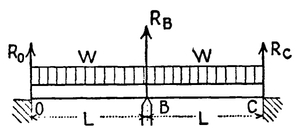


FIG. 292.

$$d_1 = \frac{5}{384}\frac{(2W)(2L)^3}{EI} \text{ (Art. 180).}$$

If the reaction R_B is assumed to act alone, the upward deflection of B would be

$$d_2 = \frac{R_B(2L)^3}{48EI} \text{ (Art. 179).}$$

Since the point B does not move, these deflections are equal. Hence

$$\frac{R_B(2L)^3}{48EI} = \frac{5}{384}\frac{(2W)(2L)^3}{EI}.$$

Therefore

$$R_B = \tfrac{10}{8}W.$$

This agrees with the result obtained in Art. 188.

Example III. Fig. 293.—Beam C is fixed at B and the other end rests on beam D. Beam D rests on two end supports. Point O is the middle of beam D. Before the load W was applied, both beams were horizontal and just touching at O. Find the pressure P between the two beams at O after the load W has been applied. Let E_1, I_1, and L_1 refer to beam C; and E_2, I_2, and L_2 to beam D.

Consider beam C. The deflection of the point O due to W and P is

$$d_1 = \frac{1}{8}\frac{WL_1^3}{E_1 I_1} - \frac{1}{3}\frac{PL_1^3}{E_1 I_1}.$$

Consider now beam D. The deflection of O due to P is

$$d_2 = \frac{1}{48}\frac{PL_2^3}{E_2I_2}.$$

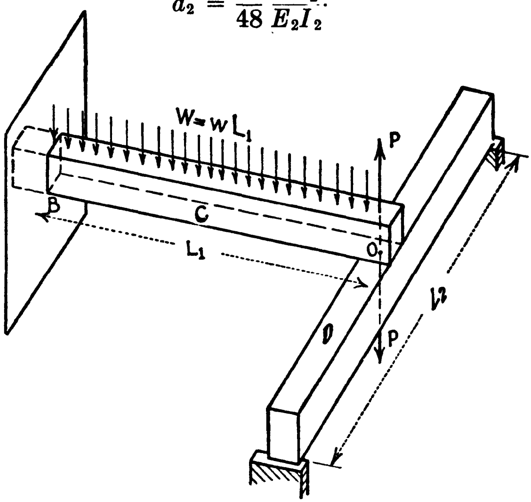


FIG. 293.

The two deflections are equal. Hence

$$\frac{1}{8}\frac{WL_1^3}{E_1I_1} - \frac{1}{3}\frac{PL_1^3}{E_1I_1} = \frac{1}{48}\frac{PL_2^3}{E_2I_2}.$$

The only unknown in the last equation is P. If $E_1 = E_2$, $L_1 = L_2$, and $I_1 = 2I_2$, the last equation becomes,

$$\frac{1}{8}W - \frac{1}{3}P = \frac{1}{24}P.$$

Or

$$P = \tfrac{1}{3}W.$$

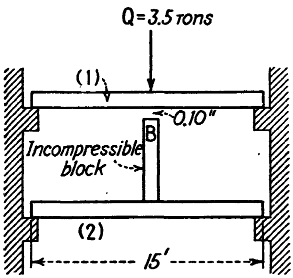


FIG. 294.

PROBLEMS

162. Given two beams as shown in Fig. 294. Initially there is a gap of $\frac{1}{10}$ in. between beam 1 and the incompressible block B. A central load of 3.5 tons is gradually applied to beam 1. The beams are 7-in. beams. I = 42 in.^{4} for each beam. Length of beams = 15 ft. E = 15,000 tons/sq. in. Find the final pressure P exerted by the upper beam (beam 1) on the block. Also find the deflection of the lower beam (beam 2).

Hint: $d_2 = d_1 - \frac{1}{10}$. *Ans.* P = 1.49 tons. d_2 = 0.2875 in.

163. A 10-in. 40-lb. standard I-beam, 30 ft. long, rests on end supports. It carries two loads, P . . . P (each 2 tons), at the one-third points of the span. Find the radius of curvature at the center; at the $\frac{1}{3}$ points. Neglect weight of beam. *Ans.* ρ = 823 ft.

164. A bar of high carbon steel, 1 in. wide and $\frac{3}{8}$ in. thick, is to be bent into the form of a circular arc. The stress in the outer fibers is not to exceed 60,000 lb./sq. in. Find the minimum radius of the circle.

Ans. ρ = 7.81 ft.

165. A 12-in. 50-lb. standard I-beam is 30 ft. long and rests on end supports. It is loaded to its full capacity at 18,000 lb./sq. in. The load is uniformly distributed. Find the deflection at the middle. E = 30,000,000 lb./sq. in.

Ans. W = 20,120 lb. (including the weight of the beam); d = 1.35 in.

166. In Problem 165, find the deflection 10 ft. from the end.

Ans. d = 1.17 in.

167. In Problem 165, find the radius of curvature of the elastic curve at the middle of the beam. *Ans.* ρ = 10,000 in.

168. The beam of Problem 165 is not to deflect more than $\frac{1}{360}$ of its length (the limit usually given for plastered ceilings). Find W. Compare the result with that of Problem 165. What is the maximum fiber stress induced in the beam? *Ans.* W = 14,890 lb. (including weight of beam).

169. In Problem 165, assume that the ends of the beam are built in horizontally. Find W and d. Compare with the results of Problem 165.

Ans. W = 30,180 lb.; d = 0.405 in.

170. A simple beam on end supports carries a uniformly distributed load W. Show that the maximum slope of the elastic curve is

$$\frac{dy}{dx} = \frac{WL^2}{24EI}.$$

What is the slope at the $\frac{1}{3}$ points?

$$\frac{dy}{dx} = \frac{13WL^2}{648EI}.$$

171. In Problem 165, what is the angle which the tangent to the elastic curve makes with the horizontal at the supports? At the $\frac{1}{3}$ points?

Ans. tan θ = 0.012; θ = 41′; tan θ = 0.00578, θ = 20′.

Note.—In Art. 176, it was stated that, since $(dy/dx)^2$ is small when compared with unity, it may be neglected in the calculus expression for $1/\rho$ [Eq. (a)]. A rigorous analysis shows that we may put $1/\rho = d^2y/dx^2$ even if dy/dx is considerably greater than the value found for the beam of Problem 171 above.

172. A steel bar, 1 in. square and 4 ft. long, is used as a cantilever. A load of 30 lb. is hung from the free end O. The deflection of O is 0.444 in. Find the modulus of elasticity. *Ans.* E = 29,890,000 lb./sq. in.

173. A steel rod 1.5 in. in diameter is built in at one end, the other end resting on a support. Both supports are at the same level. What is the maximum length this rod may have if, owing to its own weight, the maximum fiber stress is not to exceed 20,000 lb./sq. in.? Steel weighs 490 lb./cu. ft.

Ans. 76 ft. 9 in.

174. A 4- by 12-in. wooden beam is 6 ft. long. One end is built in horizontally. The other end is supported by a vertical rod 0.5 in. in diameter and 30 ft. long. The beam carries a uniformly distributed load W = 12,600

lb. For the steel, $E = 30{,}000{,}000$ lb./sq. in.; for wood, $E = 1{,}500{,}000$ lb./sq. in. Find the tension P in the rod (see Fig. 295).

Ans. $P = 3{,}320$ lb.

175. Construct the shear and moment diagrams for the beam of Problem 174. $P = 3{,}320$ lb. Compute the maximum moment and the maximum fiber stress induced in the beam.

Ans. $M_m = 17{,}880$ ft.-lb.; $S = 2236$ lb./sq. in.

176. If the right end of the beam of Problem 174 were placed on a support at the same level as the left support, what would the maximum moment be?

177. A 12-in. 31-lb. standard steel I-beam, 15 ft. long, is built in horizontally at the right end B. The other end O rests on a support. Originally the supports were at the same level. The beam carries a load of $W = 10$ tons uniformly distributed. Later the left support O sinks 1 in. Find the maximum fiber stress in the beam.

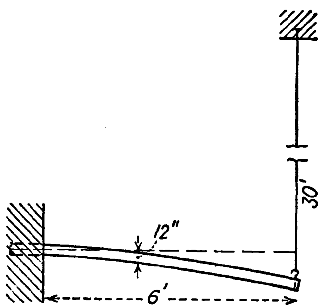


FIG. 295.

Suggestion.—Use the method of equating deflections to find the reaction at O. Then find M_m.

Ans. $R_O = 4180$ lb.; $M_m = 87{,}300$ ft.-lb.; $S = 24{,}250$ lb./sq. in.

178. In Problem 177, assume that the support O sinks 1 in. and that simultaneously the support B rotates counterclockwise through an angle of 0.6°. Find R_O. Use the method of Art. 186, being careful to read the footnote.

Ans. $R_O = 10{,}440$ lb.

179. Solve Problem 177 using the method of equating deflections.

Suggestion.—Consider the deflection of O as the combination of three deflections: (*a*) the downward deflection due to rotation of support B; (*b*) the downward deflection due to the load W; (*c*) the upward deflection due to the reaction R_O.

Ans. A local maximum moment occurs 7.83 ft. from O. $M_7 = 40{,}900$ ft.-lb.; $M_B = 6{,}690$ ft.-lb.

180. A prismatic homogeneous beam of length L is supported at the ends and has two equal loads each equal to P applied at the ends of the middle third of the span. (*a*) Using EI/ρ, find the kind of curve into which the middle third of the beam is bent. (*b*) Using $EI\ d^2y/dx^2$ and an origin at the middle of the span, find the equation of the elastic curve and compare the nature of the curve with that obtained under (*a*). Compare the results for a wooden beam, 4 in. wide by 8 in. and 15 ft. long, if each of the loads is 1000 lb. Explain the reason for the difference between the two results.

CHAPTER IX

SLOPE AND DEFLECTION. MOMENT AREA METHOD. THEOREM OF THREE MOMENTS

195. Introduction.—In Chap. VIII, a method was derived for finding the deflection of a beam and the reactions of a statically indeterminate beam. The method there used requires the integration of the differential equation

$$EI\frac{d^2y}{dx^2} = M. \qquad (a)$$

It was seen, however, that the integration of this equation may be complicated. The method now to be developed, *the slope deflection method*, greatly simplifies the algebraic detail of finding slopes, deflections, moments, and reactions.

The slope deflection method, in the form it will be developed in this chapter, applies *only* to beams *originally straight* but now *slightly bent* under the action of transverse forces (loads and reactions).* When use is made of the area of the moment diagram for the beam, the slope deflection method will be called the *moment area method*.

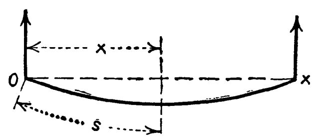


FIG. 296.

Note.—For convenience it will be assumed that originally (before the loads were applied) the beam was horizontal (and straight). The X-axis will be taken as horizontal, x being measured in the direction of the axis of the unbent beam (Fig. 296).

In the beams used for structural purposes the bending is generally so slight that no appreciable error is introduced if the x-coordinate of a point on the elastic curve is made equal to s, the distance of this point measured along the curve (see Fig. 296).

A beam frequently will be represented by its elastic curve, and *the bending will be much exaggerated.*

* Equation (a) assumes that the neutral axis is perpendicular to the plane of loading. For convenience, assume plane of loading a plane of symmetry (see Arts. 155, 158).

196. Angle between End Tangents to the Elastic Curve of a Beam.—Let AND (Fig. 297) represent the elastic curve of a beam *originally straight* and *horizontal*, but now slightly bent under the action of vertical forces. These forces are not shown and the bending is much exaggerated.

Consider the point N in this curve. In Art. 175, it is shown that, if ρ is the radius of curvature of the elastic curve and M is the bending moment in the beam at N, then, within the elastic limit,

$$\frac{EI}{\rho} = M \tag{a}$$

in which I is the moment of inertia of the section of the beam at N.

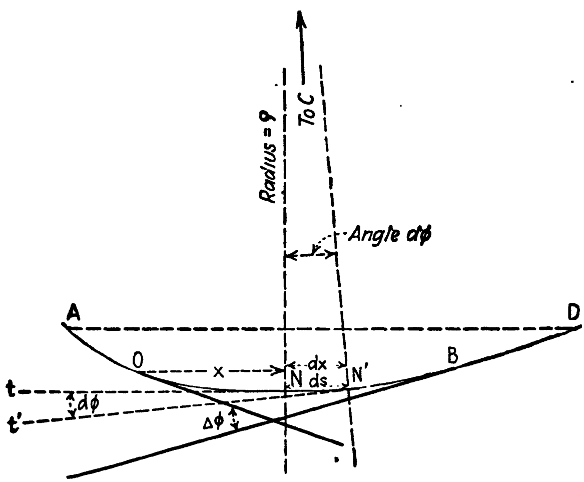


FIG. 297.

Let N' be a point in the elastic curve adjacent to the point N (Fig. 297). Bending being slight, dx may be put for ds. That is, $NN' = ds = dx$ (very nearly). Hence (from geometry)

$$dx = \rho d\phi \tag{b}$$

where $d\phi$ = angle between the two normals NC and $N'C$,
= angle between the two tangent lines tN and $t'N'$.

Eliminating ρ between Eqs. (a) and (b), we obtain as the angle between the two tangent lines tN and $t'N'$

$$d\phi = \frac{Mdx}{EI}. \tag{c}$$

Originally, before the beam was bent, $ANN'D$ was a straight line and the angle between the two tangents tN and $t'N'$ was zero. After the bending of the beam the angle between the two tangents became $d\phi$. That is, $d\phi$ equals *the change of angle between the tangents to the elastic curve for a portion* (of length dx) *of the beam.*

Let O and B (Fig. 297) be any two points in the elastic curve of the beam. Draw the tangents at O and B. Before the beam was bent the tangents at O and B coincided, but after the bending they make an angle $\Delta\phi$ with each other. That is, $\Delta\phi$ equals the *change of angle between the end tangents for the portion OB of the beam.*

If, beginning at O and ending at B, $d\phi$ is found for each dx (or ds) in succession, then the summation of the $d\phi$'s will equal $\Delta\phi$.

Or

$$\Delta\phi = \int_O^B d\phi.$$

Substituting the value $d\phi = Mdx/EI$ [Eq. (c)],

$$\Delta\phi\Big]_O^B = \int_O^B \frac{Mdx}{EI} \tag{1}$$

where $\Delta\phi\Big]_O^B$ is used to indicate that $\Delta\phi$ is the change in angle between the tangent at O and the tangent at B, *i.e.*, between the end tangents of the portion OB of the beam.

Note that $d\phi$ in Eq. (b) is measured in radians. Hence $\Delta\phi$ is measured in radians.

In the derivation of Eq. (a), M is the moment in the beam at the section N. In the summation of the $d\phi$'s for the successive dx's, N moves from O to B. Hence, to perform the integration indicated in Eq. (1), M must be expressed in terms of x, where x is the horizontal distance of N from the origin (O in this case). Bending being slight, x may be taken as the distance of N from O measured along the original or unbent axis of the beam. In a few instances, M may be constant or even zero in value throughout a part of the beam.

Example. Fig. 298.—Beam built in at B. Find change of angle between end tangents due to load P.

The moment at N is $M = Px$.

Therefore

$$\Delta\phi\Big]_O^B = \int_O^B \frac{Mdx}{EI} = \int_0^L \frac{Pxdx}{EI} = \frac{PL^2}{2EI}$$

where $\Delta\phi$ is measured in radians.

197. Relative Displacement of a Point in the Elastic Curve of a Beam.—Let O and B (Fig. 299) be two points in the elastic curve of a bent beam. *The tangent line drawn to the elastic curve at B will be called B's tangent.* In Fig. 299, $O'B$ is B's tangent. Originally before the beam was bent, O lay on B's tangent (*i.e.*, originally O' and O coincided), but now, owing to the bending of the beam, O no longer lies on B's tangent. The displacement OO' is called *the displacement of O relative to B's tangent*, or *the deflection of O from B's tangent.*

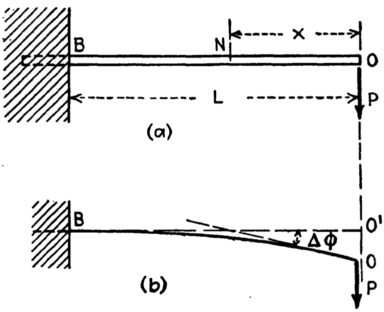


FIG. 298.

For a beam originally horizontal but now slightly bent, OO' is very nearly a vertical line so that, very nearly,

$$\Delta y\Big]_O^B = OO'$$

where $\Delta y\Big]_O^B$ *designates the vertical displacement of O relative to B's tangent* (deflection of O from B's tangent).

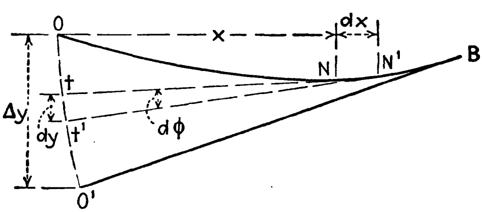


FIG. 299.

To find an expression for Δy, consider two successive points N and N' in the elastic curve. Draw tN and $t'N'$, the tangent lines at N and N', respectively. The angle between tN and $t'N'$ is [Art. 196, Eq. (*c*)]

$$d\phi = \frac{Mdx}{EI}. \qquad (a)$$

Originally, t coincided with O. Hence $tN = ON = x$ (very nearly). In the limit, as N and N' approach coincidence,

$$\overline{tt'} = \overline{tN} \times d\phi = xd\phi.$$

Or putting $tt' = dy$ and substituting the expression for $d\phi$ [Eq. (a)],

$$dy = x\frac{Mdx}{EI} = \frac{Mxdx}{EI}. \tag{b}$$

If now, beginning at O and ending at B, the dy is found for each dx in succession, then the summation of the dy's equals Δy. That is [Eq. (b)],

$$\Delta y \Big]_O^B = \int dy = \int_O^B \frac{Mxdx}{EI}. \tag{2}$$

Note again that M is the moment in the beam at N, a distance x from O. Hence to perform the integration indicated in Eq. (2) M must be expressed in terms of x.

Note also that x is measured from O whose relative displacement is desired.

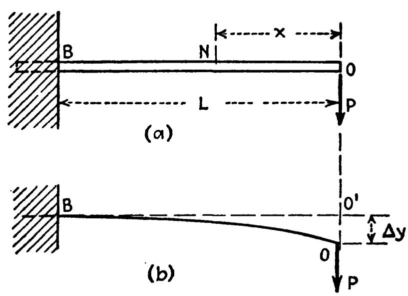


Fig. 300.

Example. Fig. 300.—Find the deflection of O from B's tangent.

$$\Delta y \Big]_O^B = \int_O^B \frac{Mxdx}{EI} = \int^L \frac{Px \times xdx}{EI} = \frac{PL^3}{3EI}.$$

Note that the tangent at B does not change in position or direction. Hence (in this case) Δy gives the actual deflection of O.

198. Résumé.—Formulas (1) and (2) may be applied to any beam that was originally straight but afterward was slightly bent under the action of transverse forces that lie in a plane that is perpendicular to the neutral axis of every section of the beam.*

* See footnote of Art. 195.

Hence, in general, O and B being any two points in the elastic curve of the beam and x being measured along the axis of the unbent beam (from point O),

$$\textit{Change of angle between tangents at O and B} = \Delta\phi\Big]_O^B = \int_O^B \frac{Mdx}{EI}. \tag{1}$$

$$\textit{Displacement of O relative to B's tangent} = \Delta y\Big]_O^B = \int_O^B \frac{Mxdx}{EI}. \tag{2}$$

Note 1.—Generally in this chapter, beams will be assumed homogeneous (E constant) and prismatic (I constant). When E and I are constants, they may be taken from under the integral sign, and formulas (1) and (2) may be written

$$\Delta\phi\Big]_O^B = \frac{1}{EI}\int_O^B Mdx \qquad \text{and} \qquad \Delta y\Big]_O^B = \frac{1}{EI}\int_O^B Mxdx.$$

Note 2.—If the beam is homogeneous (E constant) but not prismatic (I varying, as in the case of a tapering beam), formulas (1) and (2) may be written

$$\Delta\phi\Big]_O^B = \frac{1}{E}\int_O^B \frac{Mdx}{I}; \qquad \Delta y\Big]_O^B = \frac{1}{E}\int_O^B \frac{Mxdx}{I}$$

in which only the E may be taken from under the integral sign.

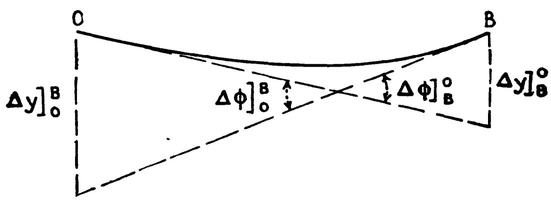


FIG. 301.

Note 3.—In Fig. 301 it is clear that the angle between the tangent lines at O and B is the same regardless of the line used as the reference line. But the displacement of O with respect to B's tangent may not be equal to the displacement of B with respect to O's tangent. That is,

$$\Delta\phi\Big]_O^B \text{ always equals } \Delta\phi\Big]_B^O.$$

But

$$\Delta y\Big]_O^B \text{ does not necessarily equal } \Delta y\Big]_B^O.$$

Note 4.—The sign of Δy in Eq. (2) depends upon the convention of sign adopted for M, the moment at a section of a beam. If the moment of an upward force is taken as positive, then $\Delta y\Big]_O^B$ will be positive if O deflects upward relative to B's tangent. As a specific illustration, in Fig. 301 $\Delta y\Big]_O^B$ is positive if the moment of upward forces (forces not shown) is taken as positive but will be negative if the moment of upward forces is taken as negative.

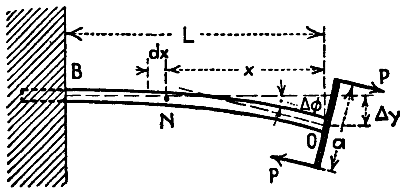


FIG. 302.

199. Unless the contrary statement is made, the following examples deal with beams which are prismatic and homogeneous, and which, before being loaded, were straight and horizontal.

Example I.—Consider a beam for which the value of the *bending moment is constant.* The sketch in Fig. 302 shows how

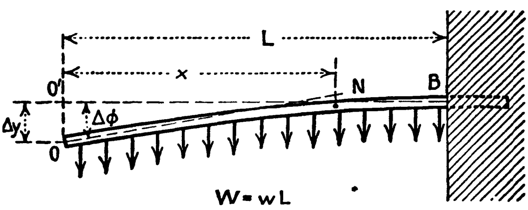


FIG. 303.

such a situation might occur. The shear is zero. The moment is $M = Pa$ (M does not vary with x).

$$\Delta\phi\Big]_O^B = \int_O^B \frac{M}{EI}dx = \frac{Pa}{EI}\int_0^L dx = \frac{PaL}{EI}$$

and

$$\Delta y\Big]_O^B = \frac{1}{EI}\int_O^B Mxdx = \frac{Pa}{EI}\int_0^L xdx = \frac{PaL^2}{2EI}.$$

Example II. *Simple Cantilever with a Uniformly Distributed Load W over the Entire Length.* Fig. 303.—Note that the tangent

line at B does not change in direction or in position. The moment at a typical section N (distant x from O) is $M = wx^2/2$, where w is load per foot of beam.

$$\Delta\phi\Big]_O^B = \frac{1}{EI}\int_O^B M dx = \frac{1}{EI}\int_0^L \frac{wx^2 dx}{2} = \frac{wL^3}{6EI} = \frac{WL^2}{6EI};$$

$$\Delta y\Big]_O^B = \frac{1}{EI}\int_O^B Mx dx = \frac{1}{EI}\int_0^L \frac{wx^3 dx}{2} = \frac{wL^4}{8EI} = \frac{WL^3}{8EI} \text{ (see Art. 182).}$$

Example III. *Simple Beam on End Supports, with Concentrated Load P at Mid Span.* Fig. 304.—Find the deflection d of

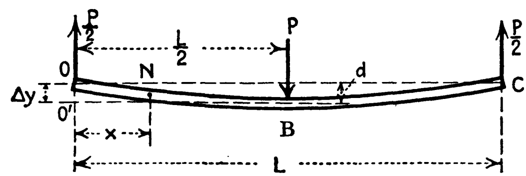


FIG. 304.

mid-point B. The tangent line at B does not change its direction. It may even be assumed that the beam is loaded while supported on level ground, and that the ends are jacked up until the ground at B exerts no pressure. The portion OB is then like a cantilever built in horizontally at B. Since the tangent line of the elastic

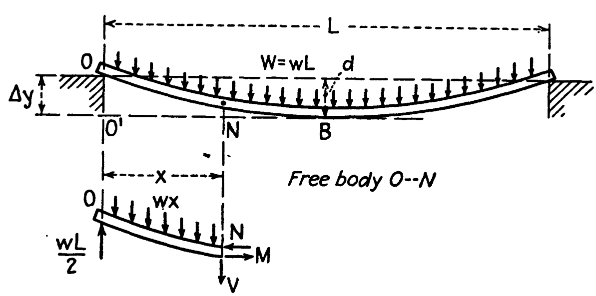


FIG. 305.

curve at mid-point B remains horizontal, the actual deflection of B equals the deflection of O relative to B's tangent.

$$d = OO' = \Delta y\Big]_O^B.$$

Let N be any point in the elastic curve of the beam between O and B. The moment at section N is $M = Px/2$.

Therefore

$$d = \Delta y\Big]_O^B = \frac{1}{EI}\int_O^B Mxdx = \frac{1}{EI}\int_0^{\frac{L}{2}}\frac{Px^2dx}{2} = \frac{PL^3}{48EI} \text{ (see Art. 179).}$$

The tendency is common to integrate from $x = 0$ to $x = L$, *i.e.*, from O to C. This is in error. We want the deflection of O from B's tangent. Hence we must integrate from $x = 0$ to $x = L/2$; *i.e.*, from O to B.

Example IV. *Simple Beam on End Supports with a Uniformly Distributed Load W over the Entire Span.* Fig. 305.—Find the deflection d of the mid-point B.

The moment at a section N distant x from point O is

$$M = \frac{wLx}{2} - \frac{wx^2}{2}.$$

$$d = OO' = \Delta y\Big]_O^B = \frac{1}{EI}\int_O^B Mxdx$$

$$= \frac{1}{EI}\int_0^{\frac{L}{2}}\left(\frac{wLx^2}{2} - \frac{wx^3}{2}\right)dx$$

$$= \frac{1}{EI}\left[\frac{wLx^3}{6} - \frac{wx^4}{8}\right]_0^{\frac{L}{2}}$$

$$= \frac{5}{384}\frac{wL^4}{EI} = \frac{5}{384}\frac{WL^3}{EI} \text{ (see Art. 180).}$$

Example V. *Simple Cantilever Beam with an Upward Concentrated End Load V and a Downward Concentrated Middle Load P.* Fig. 306.—Find the end deflection.

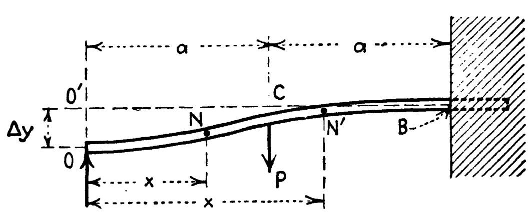


FIG. 306.

Between points O and C the expression for the moment is $M = Vx$; between C and B, $M = Vx - P(x - a)$. Hence $\frac{1}{EI}\int_O^B Mxdx$ must be divided into two parts, *viz.*, from O to C and from C to B.

Accordingly,

$$\Delta y\Big]_0^B = \frac{1}{EI}\int_0^B Mxdx = \frac{1}{EI}\int_0^a Mxdx + \frac{1}{EI}\int_a^{2a} Mxdx$$
$$= \frac{1}{EI}\int_0^a Vx^2dx + \frac{1}{EI}\int_a^{2a} [Vx - P(x-a)]xdx$$
$$= \frac{1}{EI}\frac{Vx^3}{3}\Big]_0^a + \frac{1}{EI}\Big[\frac{Vx^3}{3} - \frac{Px^3}{3} + \frac{Pax^2}{2}\Big]_a^{2a}$$
$$= \frac{(16V - 5P)a^3}{6EI} = \frac{(16V - 5P)L^3}{48EI}.$$

Example VI. *One End of a Beam Is Built in Horizontally; the Other End Rests upon a Support without Being Horizontally Constrained.* Fig. 307.—*A concentrated load P is applied at*

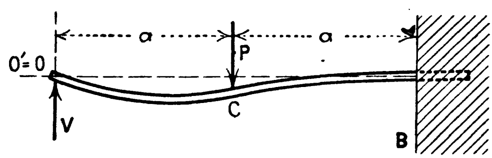


FIG. 307.

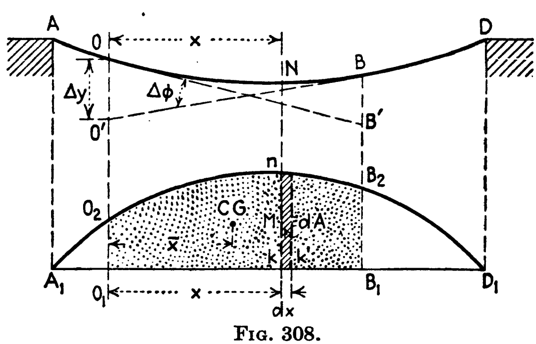


FIG. 308.

mid span. Find the reaction V at O if the end O is at the same level as B. The point O is on B's tangent. Hence the deflection of O from B's tangent is zero. That is, the value of V in Example V is such that $\Delta y = 0$. Putting $\Delta y = 0$ and solving for V,

$$V = \frac{5P}{16}.$$

Problem 181.—In Example VI, replace the concentrated load P by a load W uniformly distributed over the beam. *Ans.* $V = \frac{3}{8}W.$

200. Moment Area Method.—Let $AOBD$ (Fig. 308) be the elastic curve of a beam originally straight and horizontal. Assume the beam homogeneous and of constant cross-section so that E and I are constant.

The change in angle between tangents at O and B is [Eq. (1), Art. 198]

$$\Delta\phi = \frac{1}{EI}\int_O^B Mdx. \tag{a}$$

Let $A_1O_2B_2D_1$ be the moment diagram for this beam and its loading (loading not shown). Since M, the bending moment at a point N in the beam, is represented to scale by the corresponding ordinate nk, the product Mdx is represented by the area of the narrow strip nk'. Hence $\int_O^B Mdx$ represents the area of the moment diagram between O and B, *i.e.*, between the ordinates O_1O_2 and B_1B_2. The area of the moment diagram between O and B will be called the *moment area* between O and B and will be designated by $A\Big]_O^B$. That is

$$\int_O^B Mdx = A\Big]_O^B.$$

Equation (*a*) may be written, therefore,

$$\Delta\phi\Big]_O^B = \frac{A\Big]_O^B}{EI}. \tag{3}$$

Hence, E and I being constant, *the change in angle between the tangent at O and the tangent at B equals the moment area (between O and B), divided by EI.*

The displacement of O relative to B's tangent is [Eq. (2) Art. 198]

$$\Delta y\Big]_O^B = \frac{1}{EI}\int_O^B Mxdx. \tag{b}$$

Now $Mxdx$ may be written $x(Mdx) = xdA$, where $dA = Mdx =$ area of the narrow strip nk'. Furthermore, by the theory of center of gravity, $\int xdA = \bar{x}A$, where $\bar{x}$ equals the horizontal distance of the center of gravity of the area A from the origin. That is,

$$\int_O^B Mxdx = \int_O^B x(Mdx) = \int_O^B xdA = A\bar{x}\Big]_O^B.$$

Equation (*b*) may be written, therefore,

$$\Delta y\Big]_O^B = \frac{A\bar{x}}{EI}\Big]_O^B. \tag{4}$$

Hence, E and I being constants, *the displacement of O relative to B's tangent equals the moment of the moment area between O and B (with respect to the ordinate through O) divided by EI.*

The two formulas

$$\Delta\phi\Big]_O^B = \frac{A}{EI}\Big]_O^B \tag{3}$$

and

$$\Delta y\Big]_O^B = \frac{A\bar{x}}{EI}\Big]_O^B \tag{4}$$

are very important and therefore should be clearly understood.

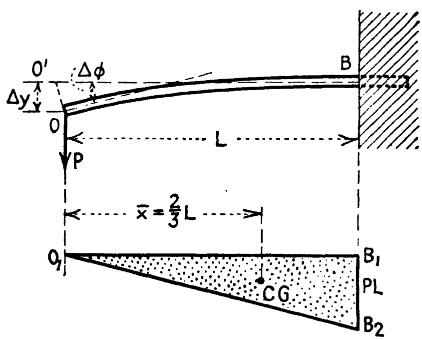


FIG. 309.

Note.—When there is no danger of ambiguity, the bracket $\Big]_O^B$ may be omitted.

Example I. *A Simple Cantilever Carries a Concentrated Load at the Free End.* Fig. 309.—The moment at B is $M = PL$. The moment diagram is a *triangle* whose base is L and whose altitude is PL. The moment area under OB is

$$A\Big]_O^B = PL \cdot \frac{L}{2} = \frac{PL^2}{2}.$$

Also

$$\bar{x} = \frac{2L}{3}.$$

Therefore, by formulas (3) and (4),

$$\Delta\phi\Big]_O^B = \frac{A}{EI} = \frac{PL^2}{2EI}$$

and

$$\Delta y\Big]_O^B = \frac{A\bar{x}}{EI} = \frac{PL^2}{2} \times \frac{2L}{3} \div EI = \frac{PL^3}{3EI}.$$

Example II. *A Simple Beam on End Supports Carries a Uniformly Distributed Load over the Entire Length.* Fig. 310.—Find the deflection d of its middle point B.

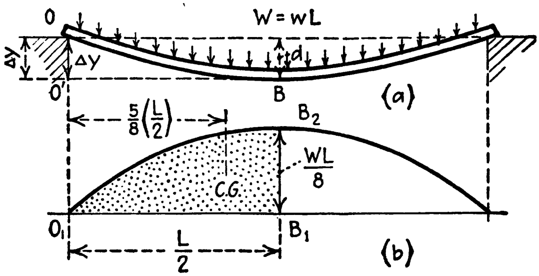


FIG. 310.

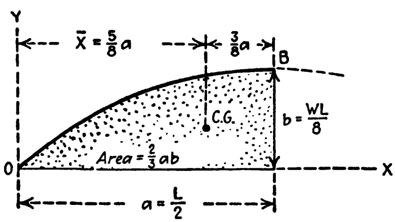


FIG. 311.

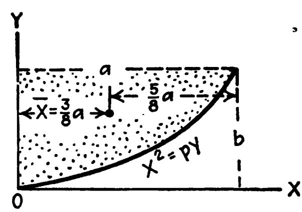


FIG. 312.

Note that $d = OO' = \Delta y\Big]_O^B$. Now Δy equals the deflection of O from B's tangent. Hence the moment area to be considered is that under OB (Fig. 310*b*), *i.e.*, the shaded portion.

The moment at B is $M = WL/8$. Therefore,*

$$A\Big]_O^B = \frac{2}{3} \times \frac{WL}{8} \times \frac{L}{2} = \frac{WL^2}{24}.$$

Also

$$\bar{x} = \frac{5}{8} \times \frac{L}{2} = \frac{5L}{16}.$$

* The shaded area (Fig. 311) is the segment of a parabola with B as its vertex. The area of the segment of the parabola is $A = \frac{2}{3}ab$. The $\bar{x}$ of the shaded area is $\bar{x} = \frac{5}{8}a$. In Example II, $a = L/2$ and $b = WL/8$.

Problem 182.—Take the equation of the parabola $x^2 = py$ (Fig. 312). Show that the area of the shaded portion is $A = \frac{2}{3}ab$, and that $\bar{x} = \frac{3}{8}a$.

Therefore

$$d = \Delta y\Big]_O^B = \frac{A\bar{x}}{EI} = \frac{WL^2}{24} \times \frac{5L}{16} \div EI = \frac{5}{384}\frac{WL^3}{EI} = \frac{5}{384}\frac{wL^4}{EI}.$$

Example III. *A Simple Beam on End Supports Carries Two Equal Concentrated Loads Symmetrically Placed.* Fig. 313.—

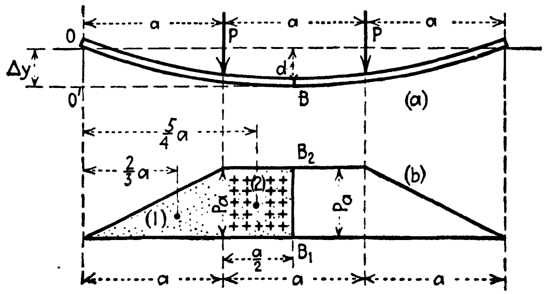


FIG. 313.

Owing to symmetry, the tangent at B remains horizontal as the beam bends. Hence

$$d = \Delta y\Big]_O^B = \frac{A\bar{x}}{EI}\Big]_O^B.$$

Draw the moment diagram (Fig. 313*b*). Since Δy is the deflection of O from B's tangent, the shaded portion only is to be considered. The shaded portion consists of two parts, a triangle and a rectangle. Using the principle that $A\bar{x}$ of a composite area equals $\Sigma(A\bar{x})$ of the separate areas, equals $A_1\bar{x}_1 + A_2\bar{x}_2$,

$$A\bar{x}\Big]_O^B = Pa \cdot \frac{a}{2} \times \frac{2}{3}a + Pa \cdot \frac{a}{2} \times \frac{5}{4}a = \frac{23}{24}Pa^3.$$

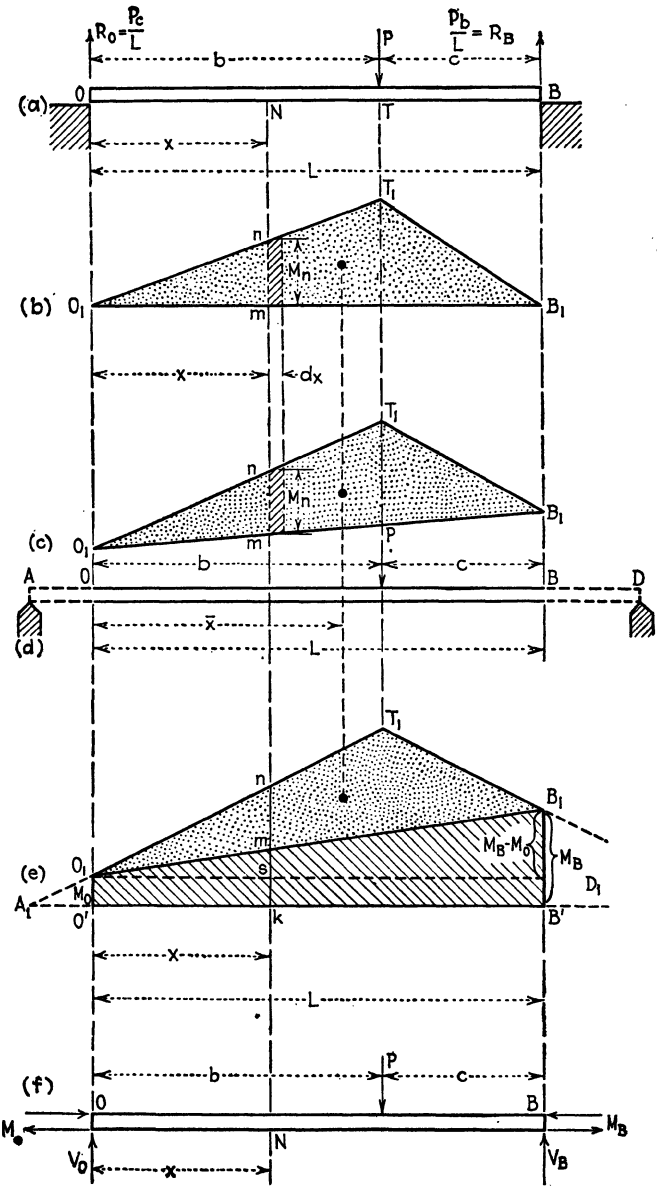
Therefore

$$d = \Delta y\Big]_O^B = \frac{A\bar{x}}{EI}\Big]_O^B = \frac{23}{24}\frac{Pa^3}{EI} = \frac{23PL^3}{648EI}.$$

201. Normal Moment Diagram.—Let OB be a simple beam on end supports (Fig. 314*a*). For simplicity the beam is assumed to carry a single concentrated load P. The results obtained hold, however, for any system of vertical loads.

The reactions of the supports are

$$R_O = \frac{Pc}{L} \quad \text{and} \quad R_B = \frac{Pb}{L}.$$



$R_0 = \frac{Pc}{L}$
P
$\frac{Pb}{L} = R_B$
b
c
(a)
0
B
N
T
x
L
T_1
n
M_n
(b)
O_1
m
B_1
x
dx
T_1
n
M_n
B_1
(c)
O_1
m
P
b
c
A
0
B
D
$\bar{x}$
(d)
L
T_1
n
B_1
m
M_B-M_0
M_B
(e)
O_1
s
D_1
A_1
M_0
O'
k
B'
x
L
P
b
c
(f)
0
B
M_0
M_B
N
V_0
x
V_B

FIG. 314.

Designating the moment at the point N by M_n,

$$M_n = \frac{Pcx}{L}. \tag{a}$$

This moment is represented in the moment diagram by the ordinate mn.

With reference to Fig. 314*b*, it should be noted that the line O_1B_1 need not be drawn horizontally. For instance, $O_1T_1B_1$ (Fig. 314*c*) is the moment diagram for OB and its loading with O_1B_1 drawn obliquely. Each ordinate mn of Fig. 314*b* equals the corresponding ordinate mn of Fig. 314*c* and each equals M_n.

In Fig. 314*a*, OB represents a simple beam on end supports. Now assume OB to be part of a longer beam $AOBD$ (Fig. 314*d*). Let $A_1T_1D_1$ (Fig. 314*e*) be the moment diagram for AD and its loading. Consider the shaded moment area $O'O_1T_1B_1B'$ under the part OB of this beam. The ordinate $O'O_1$ represents M_O, the moment in the beam at O; and $B'B_1$ represents M_B. Join O_1 and B_1. Note that the line O_1B_1 divides the moment area under OB into two parts, a dot shaded area and a line shaded area. It will be shown in Art. 202 that the part $O_1T_1B_1$ above the line O_1B_1 (the dot shaded area) is identical with the area of $O_1T_1B_1$ of Fig. 314*c* in size and in position of centroid. That is, if the beam AD be conceived as cut at O and at B, and the part OB placed on end supports (Fig. 314*a*), the moment diagram for the simple beam OB will be given by the dot shaded area $O_1T_1B_1$ of Fig. 314*e*.

In this chapter, it frequently will be necessary to construct the moment diagram for a part OB of a longer beam. For convenience, the part above the line O_1B_1 (the dot shaded area of Fig. 314*e*) will be called *the normal moment diagram for OB and its loading* for the reason that it is the moment diagram which would be obtained if OB were a simple beam on end supports (the normal case).

OB may be any part of a longer beam. To avoid the necessity of representing the longer beam, OB may be shown as a free body. With OB as a free body, the moment and shear at O and at B must be represented as is done in Fig. 314*f*.

Given M_O (the moment in the beam at O) and M_B (the moment in the beam at B), the *complete moment diagram* for OB and its loading may be constructed as follows: Draw the ordinates

$O'O_1$ and $B'B_1$ to represent the moments M_O and M_B, respectively (Fig. 314*e*). On O_1B_1 construct the *normal moment diagram* for OB and its loading.

Example. Fig. 315.—Construct the complete moment diagram for OB and its loading, assuming M_O equal to 6 ft.-tons, and M_B equal to 4 ft.-tons.

Draw $O'O_1$ to represent $M_O = 6$ ft.-tons and $B'B_1$ to represent $M_B = 4$ ft.-tons (Fig. 315*b*). On O_1B_1 as the base line, draw the normal moment diagram $O_1T_1T_2B_1$.

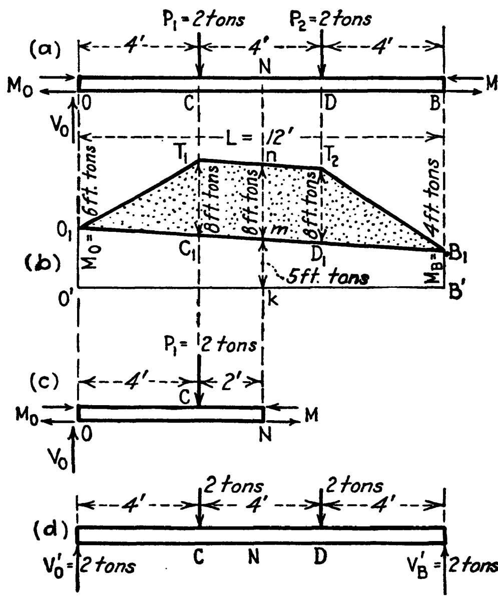


FIG. 315.

To construct the normal moment diagram for OB, treat OB as a simple beam on end supports. If OB were actually a simple beam on end supports, the moment at O and the moment at B would be zero, and the reaction at O (Fig. 315*d*) would be

$$V'_O = 2 \text{ tons.}$$

Hence at C the normal moment is

$$M_n = 2 \times 4 = 8 \text{ ft.-tons.}$$

Similarly,

$$M_n \text{ at } D = 8 \text{ ft.-tons.}$$

The complete moment M at a section N of the beam is represented by the full ordinate kn of the moment diagram (Fig. 315*b*). If N is at the middle of the beam, $km = (6 + 4)/2 = 5$ and $mn = 8$.
Therefore

$$M_N = km + mn = 5 + 8 = 13 \text{ ft.-tons.}$$

The same result for M_N may be obtained if ON is taken free (Fig. 315*c*) and the moment at N is found in the usual way. Note, however, that now it is first necessary to find V_O, the shear at O. Considering the whole beam free (Fig. 315*a*), and putting Σ moments about B equal to zero,

$$M_O - M_B + V_O \times 12 - P_1 \times 8 - P_2 \times 4 = 0.$$

With $M_O = 6$, $M_B = 4$, and $P_1 = P_2 = 2$, $V_O = 11/6$ tons
Hence (Fig. 315*c*)

$$M_N = M_O + V_O \times 6 - P_1 \times 2 = 6 + \frac{11}{6} \times 6 - 2 \times 2$$
$$= 13 \text{ ft.-tons.}$$

202. Proof of the Theorem Concerning the Normal Moment Diagram.—Let OB (Fig. 316*a*) be part of a longer beam. Assume

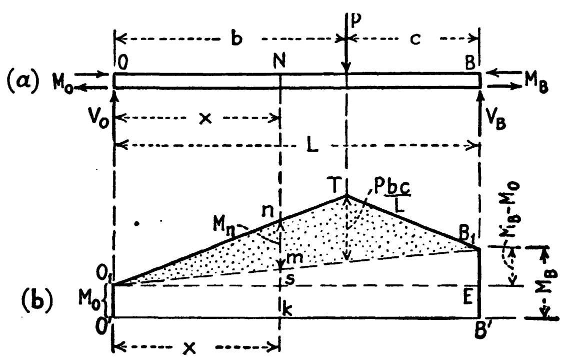


FIG. 316.

that the part OB carries a single concentrated load P. Take OB free, representing the moment and the shear at O by M_O and V_O, respectively, and those at B by M_B and V_B.

The complete moment at N is (Fig. 316*a*)

$$M = M_O + V_O x. \tag{a}$$

To find an expression for V_O, put the summation of moments about B equal to zero.

$$M_O - M_B + V_O L - Pc = 0.$$

Solving for V_O,

$$V_O = \frac{M_B - M_O}{L} + \frac{Pc}{L}.$$

Substituting in Eq. (*a*),

$$M = M_O + \frac{x}{L}(M_B - M_O) + \frac{Pcx}{L}. \tag{b}$$

Putting $M_O = 0$ and $M_B = 0$ [Eq. (*b*)] gives the normal moment at N.*
That is,

$$M_n = \frac{Pcx}{L} \text{ [see Eq. (a) Art. 201].} \tag{c}$$

Equation (*b*) may be written, therefore,

$$M = M_O + \frac{x}{L}(M_B - M_O) + M_n. \tag{d}$$

Hence M, the complete moment at N, consists of three parts [Eq. (*d*)]. The first part M_O is represented in the moment diagram (Fig. 316*b*) by *ks* (O_1E is drawn parallel to $O'B'$). The second part is represented by *sm*. This may be proved as follows:

From the similar triangles O_1sm and O_1EB_1,

$$\frac{sm}{\overline{EB}_1} = \frac{x}{L}.$$

Also

$$\overline{EB}_1 = M_B - M_O.$$

Combining

$$\overline{sm} = \frac{x}{L}(M_B - M_O).$$

The third part M_n, the normal moment, must be represented, therefore, by *mn*. That is, O_1TB_1, the part of the moment diagram above the line O_1B_1, is the normal moment diagram for OB and its loading.

* If $M_O = 0$ and $M_B = 0$, the beam is a simple beam or may be analyzed as a simple beam on end supports.

Note.—If the point N is taken to the right of the load, or if a system of loads is applied, the expression for M_n [Eq. (*c*)] becomes more complicated. Equation (*d*), however, holds for any system of loading.

203. Theorem of Three Moments.—Let AD (Fig. 317*a*) represent a homogeneous prismatic beam, originally straight and horizontal but now slightly bent under its vertical loading. Let OBC (Fig. 317*b*) represent the elastic curve of the portion OBC of this beam, showing the bending much exaggerated. Let M_O, M_B, and M_C, respectively, be the moments in the beam

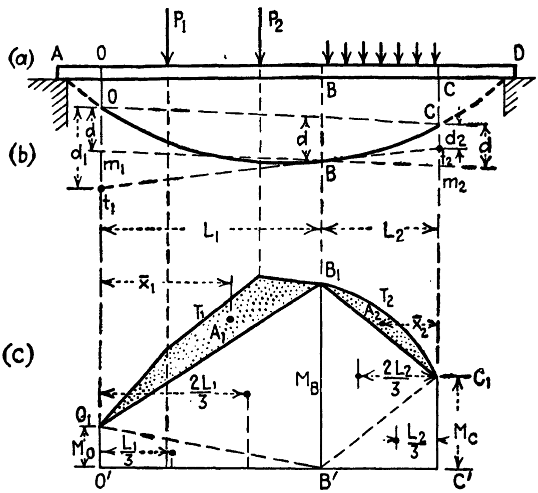


FIG. 317.

at O, B, and C. Note that M_O, M_B, and M_C are assumed to be positive (compression above and tension below, Art. 120). Construct the complete moment diagram for OBC and its loading (Fig. 317*c*), proceeding as follows: Draw a horizontal line $O'B'C'$. Draw verticals $O'O_1$, $B'B_1$, and $C'C_1$ to represent M_O, M_B, and M_C, respectively. Draw the straight lines O_1B_1 and B_1C_1. On O_1B_1 construct the normal moment diagram $O_1T_1B_1$ for OB and its loading; and on B_1C_1 construct the normal moment diagram $B_1T_2C_1$ for BC and its loading. Then $O'O_1T_1B_1T_2C_1C'$ is the complete moment diagram for OBC and its loading.

Join O_1 and B'; also B' and C_1 (Fig. 317*c*). The complete moment diagram under OB consists now of three parts, two triangles and the normal moment diagram $O_1T_1B_1$ whose area is A_1 and whose center of gravity is at a distance $\bar{x}_1$ from $O'O_1$. In

like manner, the complete moment diagram under BC consists of three parts, two triangles and the normal moment diagram $B_1T_2C_1$ whose area is A_2 and whose center of gravity lies at a distance $\bar{x}_2$ from $C'C_1$.

At the point B of the elastic curve (Fig. 317*b*), draw the tangent $t_1 \ldots t_2$. Now $\overline{Ot_1}$ (or d_1) $= \Delta y\Big]_O^B$, equals the deflection of point O from B's tangent, and hence [Art. 200, formula (4)]

$$EId_1 = A\bar{x}\Big]_O^B = \Sigma(A\bar{x})$$

for the three separate areas constituting the moment diagram under OB. That is (Fig. 317*c*),

$$EId_1 = \frac{M_OL_1}{2} \times \frac{L_1}{3} + \frac{M_BL_1}{2} \times \frac{2}{3}L_1 + A_1\bar{x}_1.$$

Or, dividing through by L_1 and simplifying,

$$\frac{EId_1}{L_1} = \frac{M_OL_1}{6} + \frac{M_BL_1}{3} + \frac{A_1\bar{x}_1}{L_1}. \qquad (a)$$

Similarly, $\overline{Ct_2}$ (or d_2) $= \Delta y\Big]_C^B$, equals the deflection of point C from B's tangent. Hence

$$EId_2 = A\bar{x}\Big]_C^B = \Sigma(A\bar{x}) \text{ of the separate areas.}$$

That is,

$$EId_2 = \frac{M_CL_2}{2} \times \frac{L_2}{3} + \frac{M_BL_2}{2} \times \frac{2}{3}L_2 + A_2\bar{x}_2.$$

Or, dividing through by L_2 and simplifying,

$$\frac{EId_2}{L_2} = \frac{M_CL_2}{6} + \frac{M_BL_2}{3} + \frac{A_2\bar{x}_2}{L_2}. \qquad (b)$$

Adding Eqs. (*a*) and (*b*),

$$EI\left(\frac{d_1}{L_1} + \frac{d_2}{L_2}\right) = \frac{M_OL_1}{6} + \frac{M_B(L_1 + L_2)}{3} + \frac{M_CL_2}{6} + \frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2}. \qquad (c)$$

Referring to Fig. 317*b*, join O and C and let d equal the deflection of B from the straight line OC. Through B draw $m_1 \ldots m_2$

parallel to OC. From similar triangles m_1Bt_1 and m_2Bt_2,

$$\frac{\overline{m_1t_1}}{L_1} = \frac{\overline{m_2t_2}}{L_2}.$$

That is

$$\frac{d_1 - d}{L_1} = \frac{d - d_2}{L_2}.$$

The last expression may be written

$$\frac{d_1}{L_1} + \frac{d_2}{L_2} = d\left(\frac{1}{L_1} + \frac{1}{L_2}\right).$$

Or, multiplying through by EI,

$$EI\left(\frac{d_1}{L_1} + \frac{d_2}{L_2}\right) = EId\left(\frac{1}{L_1} + \frac{1}{L_2}\right). \tag{d}$$

Equating right-hand members of Eqs. (c) and (d), we obtain

$$\frac{M_OL_1}{6} + \frac{M_B(L_1 + L_2)}{3} + \frac{M_CL_2}{6} + \frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2} = EId\left(\frac{1}{L_1} + \frac{1}{L_2}\right) \tag{5}$$

which is the theorem of three moments.

Note 1.—The theorem of three moments [in the form given in Eq. (5)] is applicable only to homogeneous prismatic beams.

Note 2.—For simplicity, the part OB of length L_1 will be called the *left panel;* and BC of length L_2, the *right panel.* The points O, B, and C may be any three points in the beam.

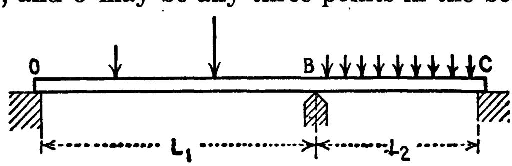


FIG. 318.

When the theorem of three moments is applied to a continuous beam, as, for example, Fig. 318, the points immediately above the three supports usually are taken as the three points O, B, and C for reasons explained later.

Note 3.—If the three points O, B, and C lie in a straight line, for instance, if the three supports (Fig. 318) are at the same level, the deflection of B from the straight line joining O and C is zero; *i.e.*, $d = 0$ and the right-hand member of Eq. (5) becomes zero.

Note 4.—If O (or C) is the end of a beam resting on a support with the end not clamped or built in, the moment at O (or C) is zero. For instance, in Fig. 318, $M_O = 0$ and $M_C = 0$.

Note 5.—In the derivation of the theorem of three moments (see Fig. 317), the bending moment at a point in the beam was considered positive if the elastic curve at that point was concave above. That is, it was assumed that a positive bending moment at a point in the beam produced compression in the upper fibers and tension in the lower fibers. If then, as a result of the application of the three moment equation, M_O or M_B or M_C is found to be negative, this indicates that the beam at O or B or C bends so that its concavity is below the beam, and that therefore the bending moment at that point produces tension above and compression below. For instance (Fig. 319), suppose that as the result of the application of the theorem of three moments it is found that $M_O = +20$ ft.-tons and that $M_B = -40$ ft.-tons. These signs show that at O the stress couple should be drawn to indicate compression above and tension below, while the stress-couple at B should be drawn to indicate tension above and compression below. *Note that when the stress-couple at B in Fig.* 319 *is drawn to indicate tension above and compression below, the negative sign for* M_B *(obtained as a result of the application of the theorem of three moments) is properly observed,* and that now

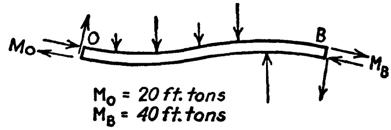


FIG. 319.

$$M_B = +40 \text{ ft.-tons.}$$

Note 6.—In the derivation of the three-moment equation, it was assumed that B deflects down from the line OC (Fig. 317). Hence, if B deflects down from OC (Fig. 320*a*) d is positive (+); if up (Fig. 320*b*), d is negative (−).

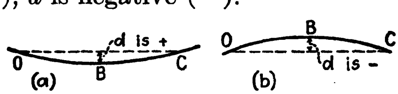


FIG. 320.

Note 7.—When a panel is not loaded, the term $A\bar{x}/L$ for the normal moment diagram for that panel is zero. For instance, if the left panel is not loaded, the area A_1 of the normal moment diagram for the left-hand panel is zero. Hence,

$$\frac{A_1\bar{x}_1}{L_1} = 0.$$

204. Values of $A_1\bar{x}_1/L_1$ and $A_2\bar{x}_2/L_2$ for Special Cases of Loading.—In the theorem of three moments, O, B, and C are

any three points in the beam (Fig. 321). For convenience, OB is called the left panel and BC the right panel. The expressions $A_1\bar{x}_1/L_1$ and $A_2\bar{x}_2/L_2$ refer, respectively, to the normal moment diagrams for the left and for the right panel, *i.e.*, for OB and BC. If, for instance, $A_1\bar{x}_1$ is to be found, OB is treated as a simple beam on end supports (Fig. 322*a*) and $A_1\bar{x}_1$ is found from its

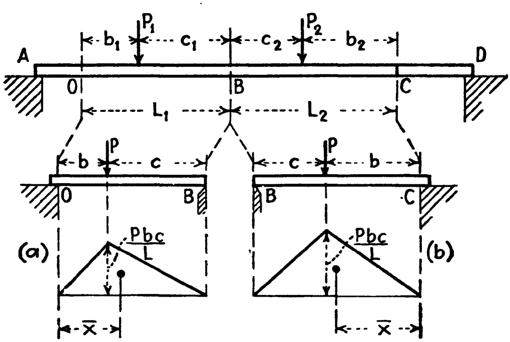


FIG. 321. FIG. 322.

moment diagram, $\bar{x}_1$ being measured from the left end O. In like manner, if $A_2\bar{x}_2$ is to be found, BC is treated as a simple beam on end supports (Fig. 322*b*), $\bar{x}_2$ being measured from the right end C.

Consider now a single concentrated load P. If P is on the left panel, let b designate its distance from the left or outer end O, and let c designate its distance from the intermediate point B. If P is on the right panel, let b and c designate its distance, respectively, from the right or outer end C and from the intermediate point B. That is, *for either panel, b is measured from the outer end of the panel.* If for any reason it is desired to designate which panel is under consideration, b_1, c_1, and L_1 will be used for the left panel and b_2, c_2, and L_2 for the right panel, as shown in Fig. 321. General expressions or formulas will now be obtained for $A\bar{x}/L$ for special cases of loading.

Note.—It sometimes will be convenient to let $H = A\bar{x}/L$, so that, for the left panel,

$$H_1 = \frac{A_1\bar{x}_1}{L_1},$$

and for the right panel,

$$H_2 = \frac{A_2\bar{x}_2}{L_2}.$$

Or, if it is desired to designate the part of the beam considered as a simple beam,

$$H_{OB} = \left.\frac{A\bar{x}}{L}\right]_O^B, \qquad H_{CB} = \left.\frac{A\bar{x}}{L}\right]_C^B.$$

Case I. *Noncentral Concentrated Load P*. Fig. 323.—For either panel, the moment under P is*

$$M = \frac{Pbc}{L} \tag{a}$$

and is represented by the ordinate NT of the moment diagram. To find an expression for $A\bar{x}/L$, either panel may be considered. Consider the left panel (Fig. 323*b*). Extend the line B_1T to meet

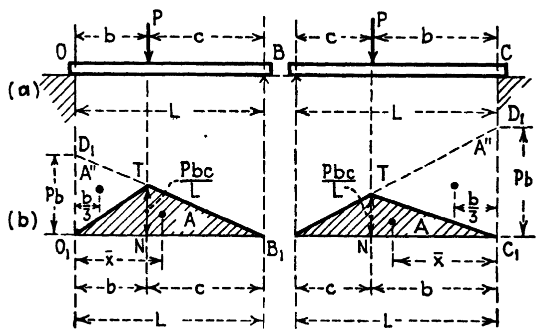


FIG. 323.

the vertical through O_1 at D_1. If the area of the triangle $O_1D_1B_1$ is designated by A', and that of triangle O_1D_1T by A'', then from the theory of center of gravity of a compound figure

$$A\bar{x} \text{ (of shaded area)} = A'\bar{x}' \text{ (of area } O_1D_1B_1) - A''\bar{x}'' \text{ (of area } O_1D_1T). \tag{b}$$

* Given a simple beam (of length L) on end supports.

1. For a central load P (Fig. 324), the moment under P is $M = PL/4$.
2. For a noncentral load (Fig. 323), the moment under P is $M = Pbc/L$.
3. For a uniformly distributed load W, the moment at the center is

$$M = \frac{WL}{8}.$$

These equations are used frequently. It is very important clearly to see and to remember them (see Art. 143).

From similar triangles ($B_1O_1D_1$ and B_1NT),

$$\frac{\overline{O_1D_1}}{\overline{NT}} = \frac{L}{c}.$$

Therefore

$$\overline{O_1D_1} = \frac{L}{c} \times \overline{NT}.$$

Or, since $\overline{NT} = Pbc/L$ [Eq. (*a*)],

$$\overline{O_1D_1} = \frac{L}{c} \times \frac{Pbc}{L} = Pb.$$

Therefore

$$A'\bar{x}' \text{ (of area } O_1D_1B_1) = \left(Pb \times \frac{L}{2}\right) \times \frac{L}{3} = \frac{PbL^2}{6}.$$

$$A''\bar{x}'' \text{ (of area } O_1D_1T) = \left(Pb \times \frac{b}{2}\right) \times \frac{b}{3} = \frac{Pb^3}{6}.$$

Hence [Eq. (*b*)]

$$A\bar{x} = \frac{PbL^2}{6} - \frac{Pb^3}{6} = \frac{Pb}{6}(L^2 - b^2) = \frac{Pb}{6}(L - b)(L + b) = \frac{Pbc(L + b)}{6}.$$

Or, divided through by L, for either panel,

$$H = \frac{A\bar{x}}{L} = \frac{Pbc(L + b)}{6L} = \frac{Pb(L^2 - b^2)}{6L} \tag{I}$$

where b is measured from the left end for a left panel, and from the right end for a right panel.

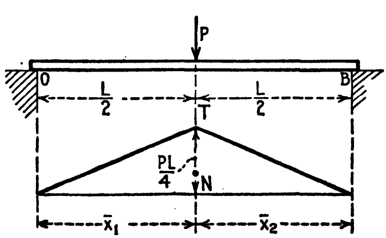


FIG. 324.

Case II. *Single Central Concentrated Load P.* Fig. 324.—For a central concentrated load, $b = c = L/2$. Using Case I and putting $b = c = L/2$, we obtain for either panel

$$H = \frac{A\bar{x}}{L} = \frac{PL^2}{16} \tag{II}$$

The same result may be obtained directly as follows. The moment under the load is $M = PL/4 = \overline{NT}$.

Therefore,

$$A = \frac{PL}{4} \times \frac{L}{2} = \frac{PL^2}{8}.$$

Also

$$\bar{x}_1 = \bar{x}_2 = \frac{L}{2}.$$

Therefore

$$\frac{A\bar{x}}{L} = \frac{PL^2}{16}.$$

Case III. *Two or More Concentrated Loads.* Fig. 325.—For convenience, assume two loads P' and P'' as shown. At any point in the beam the moment due conjointly to P' and P'' equals the sum of the moments due to P' and P'' acting separately, or $M = M' + M''$. Hence, if Fig. 325*b* is the moment diagram

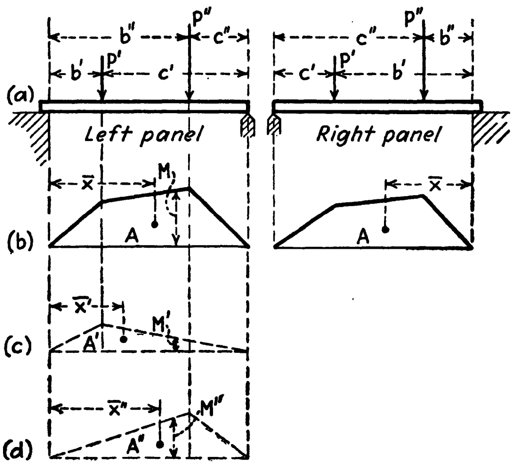


FIG. 325.

due to both loads, Fig. 325*c* that due to P' acting alone, and Fig. 325*d* that due to P'' acting alone, then each ordinate of Fig. 325*b* equals the sum of the corresponding ordinates of Figs. 325*c* and 325*d*. Therefore, $A = A' + A''$ and Fig. 325*b* may be considered to be a combination of Figs. 325*c* and 325*d*. Hence $A\bar{x} = A'\bar{x}' + A''\bar{x}''$. Dividing through by L,

$$\frac{A\bar{x}}{L} = \frac{A'\bar{x}'}{L} + \frac{A''\bar{x}''}{L} + \cdots \text{ (a like term for each additional load).}$$

That is, $A\bar{x}/L$ *of a system of concentrated loads* equals $\Sigma(A\bar{x}/L)$ of the separate loads. Or, making use of Case I,

$$H = \frac{A\bar{x}}{L} = \frac{P'b'c'(L + b')}{6L} + \frac{P''b''c''(L + b'')}{6L} + \cdots$$

$$\left.\begin{aligned} &= \sum \frac{Pbc(L + b)}{6L} \\ &= \sum \frac{Pb(L^2 - b^2)}{6L} \end{aligned}\right\} \quad \text{(III)}$$

where the b's are *measured from the left end for a left panel and from the right end for a right panel.*

Note.—From the law of superposition, formula (III) follows immediately from formula (I).

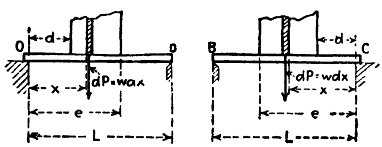


FIG. 326.

Case IV. *Continuous Load over Any Part of the Span.* Fig. 326.—Let w equal the rate of loading (load per linear foot) at a point distant x from the end. Then wdx is the load on a length dx, and this load may be treated as a concentrated load like one of the P's of Case III. Hence, using Case III, since $b = x$,

$$H = \frac{A\bar{x}}{L} = \sum \frac{Pb(L^2 - b^2)}{6L} = \int_d^e \frac{wdx\, x(L^2 - x^2)}{6L} =$$

$$\int_d^e \frac{w(L^2 - x^2)xdx}{6L}. \quad \text{(IV)}$$

Here x is measured from the left end for a left panel and from the right end for a right panel.

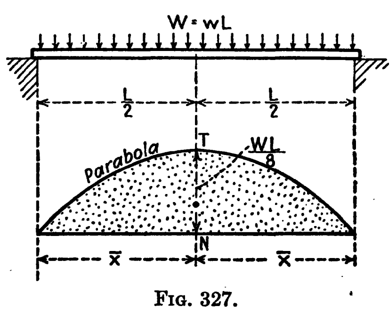


FIG. 327.

Note.—If the load is uniformly distributed, w is constant and may be taken from under the integral sign. If w is variable,

it must be expressed in terms of x before the integration can be performed.

Case V. *Uniformly Distributed Load over Whole Span.* Fig. 327.

$$W = wL.$$

The rate of loading is w and is a constant. From Case IV,

$$H = \frac{A\bar{x}}{L} = \frac{w}{6L}\int_0^L (L^2 - x^2)x dx$$
$$= \frac{w}{6L}\left[\frac{L^2x^2}{2} - \frac{x^4}{4}\right]_0^L$$
$$= \frac{wL^3}{24} = \frac{WL^2}{24}. \tag{V}$$

The value of $A\bar{x}/L$ may be found directly as follows:

$$\overline{NT} = \frac{WL}{8}, \qquad A = \tfrac{2}{3}\overline{NT} \times L, \qquad \bar{x} = \tfrac{1}{2}L.$$

Therefore

$$H = \frac{A\bar{x}}{L} = \frac{\frac{2}{3}\left(\frac{WL}{8}L\right)\frac{L}{2}}{L} = \frac{WL^2}{24} = \frac{wL^3}{24}.$$

Case VI. *Triangularly Distributed Load over the Entire Panel.* Fig. 328.—Let q equal the rate of loading at unit distance from O

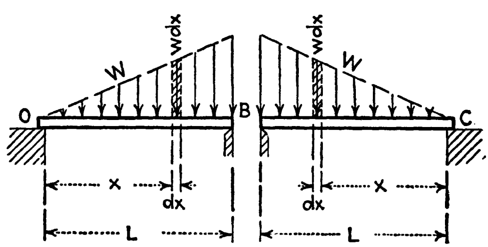


FIG. 328.

(or C). Then $w = qx$, equals the rate of loading at distance x from O (or C). The total load on the panel is

$$W = \int_0^L w dx = \int_0^L qx dx = \frac{qL^2}{2}.$$

Therefore, solving for q,

$$q = \frac{2W}{L^2}. \tag{a}$$

Using Case IV, since $w = qx$,

$$\frac{A\bar{x}}{L} = \int_0^L \frac{qx(L^2 - x^2)xdx}{6L} = \frac{q}{6L}\left[\frac{L^2x^3}{3} - \frac{x^5}{5}\right]_0^L = \frac{qL^4}{45}.$$

Or [Eq. (*a*)]

$$H = \frac{A\bar{x}}{L} = \frac{2WL^2}{45}. \qquad \text{(VI)}$$

Note that the vertex of the triangle is at O (or C). *If the vertex of the triangle is at B,*

$$H = \frac{A\bar{x}}{L} = \frac{7WL^2}{180}.$$

Case VII. *System of Loads.* Fig. 329.—By the method of reasoning used in Case III, it can be shown that for any two or more systems of loads

$$\frac{A\bar{x}}{L}\textit{ for a system of loads} = \sum\frac{A\bar{x}}{L}\textit{ for the separate loads.} \qquad \text{(VII)}$$

For instance, Fig. 329 shows a panel carrying a uniformly dis-

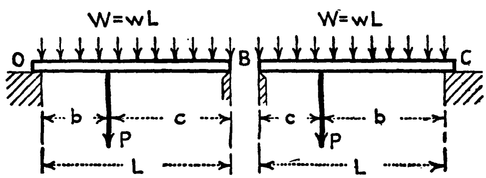


FIG. 329.

tributed load W (Case V) and a concentrated load P (Case I) Hence

$$H = \frac{A\bar{x}}{L} = \frac{WL^2}{24} + \frac{Pbc(L + b)}{6L}.$$

205. Application of the Theorem of Three Moments to a Continuous Beam on Three Supports.—In the solution of a continuous beam on three supports, it is necessary as a rule to determine the reactions. In the determination of these reactions, the theorem of three moments may be used to great advantage. *Note carefully the method of procedure used* in the solution of the following example. Note also that, with the exception of the first step, the general procedure is the same as that for a simple beam.

Example I. Fig. 330.—Given a homogeneous prismatic beam on three supports at the same level. The left panel (L_1 = 20 ft.)

carries a load of 8 tons uniformly distributed over the whole span and the right panel ($L_2 = 14$ ft.) carries a single concentrated load of 5 tons as shown. First determine M_B, the moment of the beam over the middle support, and then the reactions R_O, R_B, and R_C. Also draw the shear and moment diagrams and determine the maximum shear and the maximum moment.

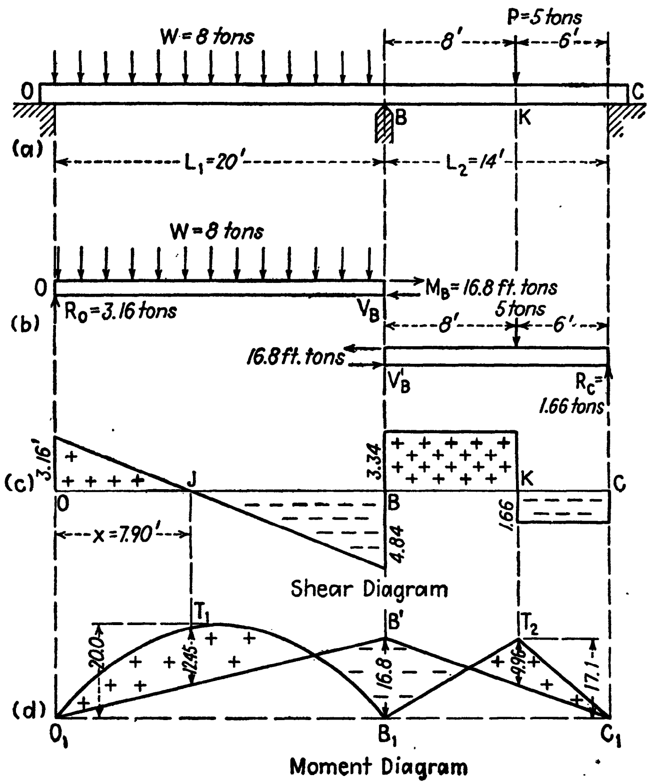


FIG. 330.

1. *To determine M_B,* apply the theorem of three moments to the three sections O, B, and C (immediately above the supports). The three moment equation is [Eq. (5) Art. 203]

$$\frac{M_O L_1}{6} + \frac{M_B(L_1 + L_2)}{3} + \frac{M_C L_2}{6} + \frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2} = EId\left(\frac{1}{L_1} + \frac{1}{L_2}\right). \quad (5)$$

Since the three supports are on the same level, $d = 0$ (Note 3, Art. 203). Hence the right-hand member of Eq. (5) becomes zero. Since the end supports are simple contact supports (Note 4, Art. 203), $M_O = 0$ and $M_C = 0$.

For the left panel (Case V),

$$\frac{A_1\bar{x}_1}{L_1} = \frac{WL^2}{24} = \frac{8(20)^2}{24} = 133.3.$$

For the right panel (Case I),

$$\frac{A_2\bar{x}_2}{L_2} = \frac{Pbc(L + b)}{6L} = \frac{5 \times 6 \times 8(14 + 6)}{6 \times 14} = 57.1.$$

Substituting these values in Eq. (5), putting $d = 0$, $M_O = 0$, $M_C = 0$,

$$0 + \frac{M_B(20 + 14)}{3} + 0 + 133.3 + 57.1 = 0.$$

Or

$$M_B = -16.80 \text{ ft.-tons.}$$

2. *Reactions.*—The negative sign of M_B indicates that at B the beam bends so that its concavity is below (tension above and compression below; see Note 5, Art. 203). Take OB free, cutting the beam just to the left of B. In Fig. 330*b* the minus sign of M_B is observed by drawing the stress-couple at B to indicate tension above and compression below.

Σ moments about $B = 0$ gives

$$20R_O - 8 \times 10 + 16.8 = 0.$$

Therefore

$$R_O = 3.16 \text{ tons.}$$

In like manner, take BC free, cutting the beam just to the right of B (Fig. 330*b*). The stress-couple is again drawn to indicate tension above and compression below.

Σ moments about $B = 0$ gives

$$14R_C - 5 \times 8 + 16.8 = 0.$$

Therefore

$$R_C = 1.66 \text{ tons.}$$

Σ vertical forces $= 0$ gives (Fig. 330*a*)

$$R_O + R_B + R_C - 5 - 8 = 0.$$

Therefore

$$R_B = 8.18 \text{ tons.}$$

3. *Shear Diagram. Maximum Shear.*—With the reactions at the supports known, the shear diagram may be drawn in the same way as for simple beams. In Fig. 330*c*, just to the right of O, the shear is $V = 3.16$ tons; just to the left of B,

$$V = 3.16 - 8 = -4.84 \text{ tons};$$

between B and K, $V = -4.84 + 8.18 = 3.34$ tons; and between K and C, $V = 3.34 - 5 = -1.66$ tons. The shear that is numerically the largest occurs just to the left of B and is

$$V_{\text{max.}} = 4.84 \text{ tons}.$$

4. *Maximum Moment* $M_{\text{max.}}$.—A local maximum occurs at every section of the beam at which the shear passes through a zero value, *i.e.*, at J, B, and K (Fig. 330*c*). At J (distant x from O) the vertical shear is zero, or

$$\begin{aligned} V_J &= R_O - wx = 0 \\ &= 3.16 - \frac{8x}{20} = 0. \end{aligned}$$

Therefore

$$x = 7.90 \text{ ft.}$$

The moment at J is

$$M_J = R_O x - \frac{wx^2}{2} = 3.16 \times 7.90 - \frac{8}{20} \cdot \frac{(7.90)^2}{2} = 12.45 \text{ ft.-tons.}^*$$

The moment at K is (see Art. 120 for rule of sign)

$$M_K = R_C \times 6 = 1.66 \times 6 = 9.96 \text{ ft.-tons.}$$

The moment at B has already been found and equals -16.8 ft.-tons. Of the three possible maximum values M_J, M_K, and M_B, M_B is numerically the largest. Hence

$$M_{\text{max.}} = 16.8 \text{ ft.-tons.}$$

5. *Moment Diagram.* Fig. 330*d*.—The moment diagram for the left panel is a parabola and that for the right panel consists of two straight lines.

On O_1B_1 construct the normal moment diagram $O_1T_1B_1$ for the left panel, and on B_1C_1 construct the normal moment diagram

* The shear area theorem may be used to advantage (Art. 125).

$$M_J = \frac{3.16 \times 7.90}{2} = 12.45 \text{ ft.-tons.}$$

$B_1T_2C_1$ for the right panel. Draw B_1B' to represent M_B. Join O_1 and B'; also B' and C_1. Intercepts in plus shaded areas give positive moments, and intercepts in minus shaded areas give negative moments.

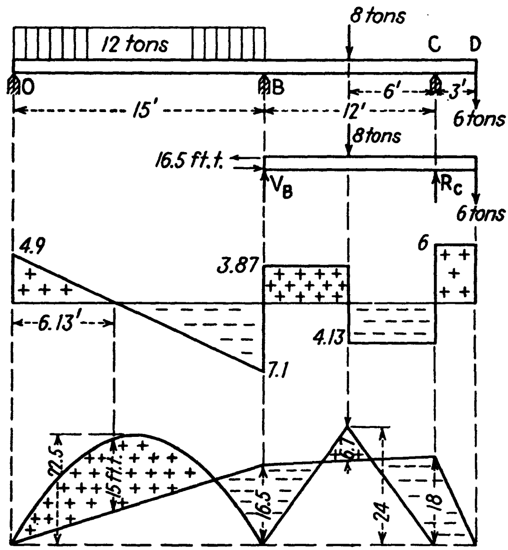


FIG. 331.

Example II. Figure 331 represents an overhanging beam on three supports at the same level. Apply the theorem of three moments to the points O, B and C and determine M_B.
Note that the moment at C is

$$M_C = -6 \times 3 = -18 \text{ ft.-tons.}$$

(the moment of a downward force is negative in accordance with our convention of sign). Hence the equation of three moments may be written

$$0 + M_B\frac{(15 + 12)}{3} - \frac{18 \times 12}{6} + \frac{12 \times 15^2}{24} + \frac{8 \times 12^2}{16} = 0$$

Therefore,

$$M_B = -16.5 \text{ ft.-tons.}$$

To find the reaction at C, take BD free;

$$R_C \times 12 + 16.5 - 8 \times 6 - 6 \times 15 = 0$$
$$R_C = 10.13 \text{ tons.}$$

Similarly $R_O = 4.9$ tons and $R_B = 10.97$ tons.

206. Selection of Economical Beam.—If it is required to select an economical beam (an I-beam in the case of Example I of Art. 205), the method of procedure is analogous to that used for a simple beam. That is, first find Z_1, the section modulus, required for the beam to enable it to carry the applied loads (weight of the beam not considered). Then select a beam whose section modulus Z is a little larger than Z_1 to take care of the weight of the beam. Consider next the effect of the weight of the beam alone and determine the section modulus Z_2 required for the beam to carry its own weight (at the same section previously used for the maximum moment due to the applied loads). Evidently Z should not be less than $Z_1 + Z_2$.

Example.—Referring to Example I of Art. 205, select an economical I-beam. Use an allowable unit stress of 15,000 lb./sq. in.

6. To determine Z_1, put $SI/c = M_m$.

$$15{,}000\frac{I}{c} = 16.80 \times 2000 \times 12 = 403{,}000 \text{ in.-lb.}$$

Therefore

$$\frac{I}{c} = 26.9 = Z_1.$$

Hence a section modulus of $Z_1 = 26.9$ in.^{3} is required to enable the beam to carry safely the applied loads.

7. Select a beam whose section modulus Z is a little greater than 26.9. Try a 12-in. standard I-beam weighing 31.8 lb./ft., whose section modulus is $Z = 36$ in.^{3} (A 10-in. I-beam weighing 35 lb./ft. has a section modulus of 29.2 in.^{3} Being heavier, it would be more expensive.)

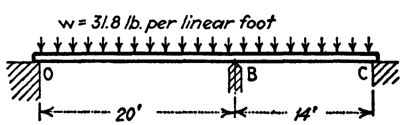


FIG. 332.

8. *Consider the weight of the beam.* Treat the beam as uniformly loaded with 31.8 lb./lin. ft. (Fig. 332). At the section of maximum moment found for the applied loads (at B in this instance) find the additional value of I/c; *i.e.*, find Z_2 required by the beam's own weight. Applying the theorem of three

moments (using Case V for both panels),

$$0 + \frac{M'_B(20 + 14)}{3} + 0 + \frac{31.8(20)^3}{24} + \frac{31.8(14)^3}{24} = 0.$$

Therefore $M'_B = -1255$ ft.-lb. Putting

$$15{,}000\frac{I}{c} = 1255 \times 12 = 15{,}060,$$

$$\frac{I}{c} = 1.00 \text{ in.}^3 = Z_2.$$

Z is larger than $Z_1 + Z_2$; *i.e.*, 36 is greater than $26.9 + 1.00$. The beam selected is large enough to carry its own weight in addition to the applied loads.*

PROBLEMS

In the following problems, the beams are homogeneous and prismatic, and the supports are at the same level. For the method of solution, see Art. 205. Verify the results given.

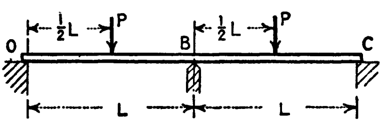


FIG. 333.

183. Fig. 333.—The two spans are of equal length L. Each span carries a central concentrated load P (Case II, Art. 204). Neglect the weight of the beam.

Use the theorem of three moments to determine M_B. Then find reactions R_O, R_B, and R_C.

Ans. $M_B = -\frac{3PL}{16}$; $R_O = \frac{5P}{16} = R_C$; $R_B = \frac{11P}{8}$; $M_m = \frac{3}{16}PL$.

184. Fig. 334.—Same as Problem 183, except that the beam carries a

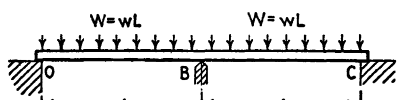


FIG. 334.

uniformly distributed load W over each span (Case V, Art. 204).

Ans. $M_B = -\frac{WL}{8}$; $R_O = R_C = \frac{3W}{8}$; $R_B = \frac{10W}{8}$; $M_m = \frac{WL}{8}$.

185. Fig. 335.—Neglect the weight of the beam and find M_B, R_O, R_C, and R_B. Also construct shear and moment diagrams.

Ans. $M_B = -15$ ft.-tons, $R_O = 2.25$ tons. $R_C = 1.56$ tons, $R_B = 7.19$ tons, $x = 7.50$ ft., $M_J = 8.44$ ft.-tons, $M_K = 12.48$ ft.-tons, and $M_m =$ 15 ft.-tons.

* If for any reason a beam of different weight w' per linear foot is to be considered, the corresponding Z'_2 may be found from the relation $Z'_2/Z_2 = w'/w$ (see Note 4, Art. 152). For example, a 10-in. 35-lb. beam would require $Z'_2 = 35/31.8 \times 1.00 = 1.11$ in.³ This beam would be safe but more expensive.

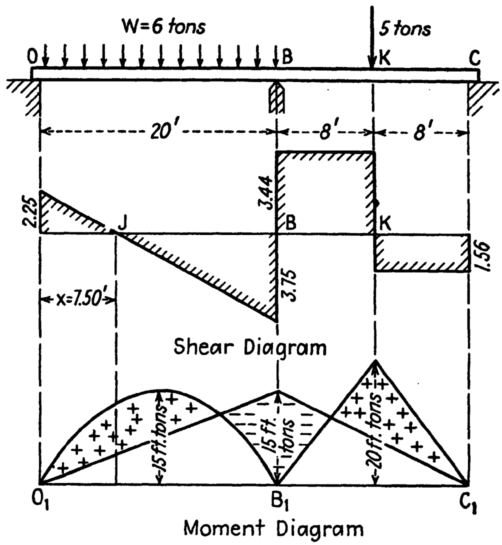


FIG. 335.

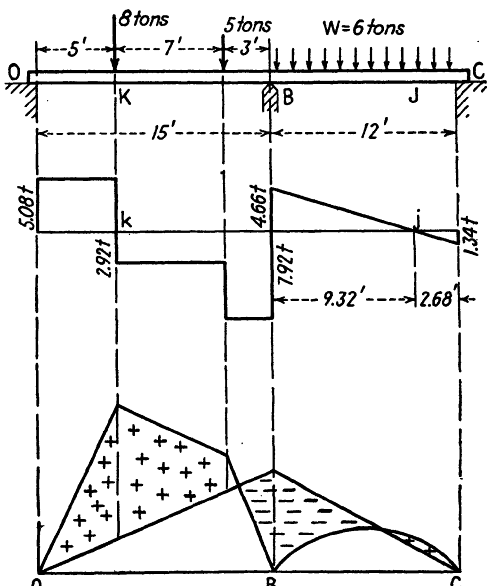


FIG. 336.

186. Select an economical standard I-beam as the beam for Problem 185. Follow the outline of Art. 206. Take S as 18,000 lb./sq. in. for steel.

The value of I/c required to carry the applied loads is $Z_1 = 20$ in.^{3} Try a 10-in. 25.4-lb. I-beam whose I/c is 24.4 in.$^3 = Z$.

Treating the beam as carrying its own weight alone, $M'_B = -1067$ ft.-lb., $Z_2 = 0.71$ in.^{3} Since $Z > Z_1 + Z_2$, the beam is satisfactory.

187. Fig. 336.—A beam with loads as shown is supported by three piers at the same level. Neglecting the weight of the beam, find M_B, R_O, R_C, and R_B. Also construct shear and moment diagrams.

Results are as follows: $M_B = -19.88$ ft.-tons, $M_K = 25.05$ ft.-tons, $M_J = 1.8$ ft.-tons, $M_m = M_K = 25.05$ ft.-tons, $R_O = 5.01$ tons, $R_C = 1.34$ tons.

188. Select an economical I-beam for the beam of Problem 187. Take $S = 18{,}000$ lb./sq. in.

Results are as follows: $Z_1 = 33.40$ in.^{3} Try a 12-in. 31.8-lb. I-beam whose section modulus is $Z = 36$ in.^{3} Treating this beam as carrying solely its own weight, $M'_B = -751$ ft.-lb., $R'_O = 188$ lb., $M'_K = 543$ ft.-lb., $Z_2 = 0.362$ in.^{3} Note that Z_2 must be found for the section at K, since the maximum moment occurs at that section.

The beam selected is satisfactory, since $Z > Z_1 + Z_2$.

Note.—An experienced designer would know that the trial beam in this case has a margin in its section modulus more than sufficient to take care of the beam's own weight.

207. Continuous Beams on More than Three Supports.—Figure 337 represents a prismatic, homogeneous continuous beam on five supports at the same level. The ends O and E rest on simple contact supports (not clamped or built in). Hence $M_O = 0$ and $M_E = 0$.

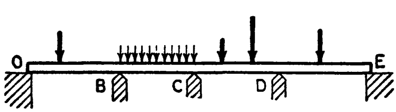


FIG. 337.

The theorem of three moments may be applied to any three points in a beam. To find M_B, M_C, and M_D apply the theorem first to the three sections O, B, and C; next to B, C, and D; and finally to C, D, and E. This procedure furnishes three simultaneous equations in M_B, M_C, and M_D. The solution of these three equations determines the three moments M_B, M_C, and M_D. An extension of this procedure will serve to make solutions of more complicated cases.

It should be remembered that, when the theorem of three moments is applied to the portion BCD, for instance, BC is the left panel and CD is the right panel.

Example. Fig. 338.—The continuous beam has four supports at the same level and is loaded as shown. Find M_B and M_C; also the reactions of the four supports, *viz.*, R_O, R_B, R_C, and R_D.

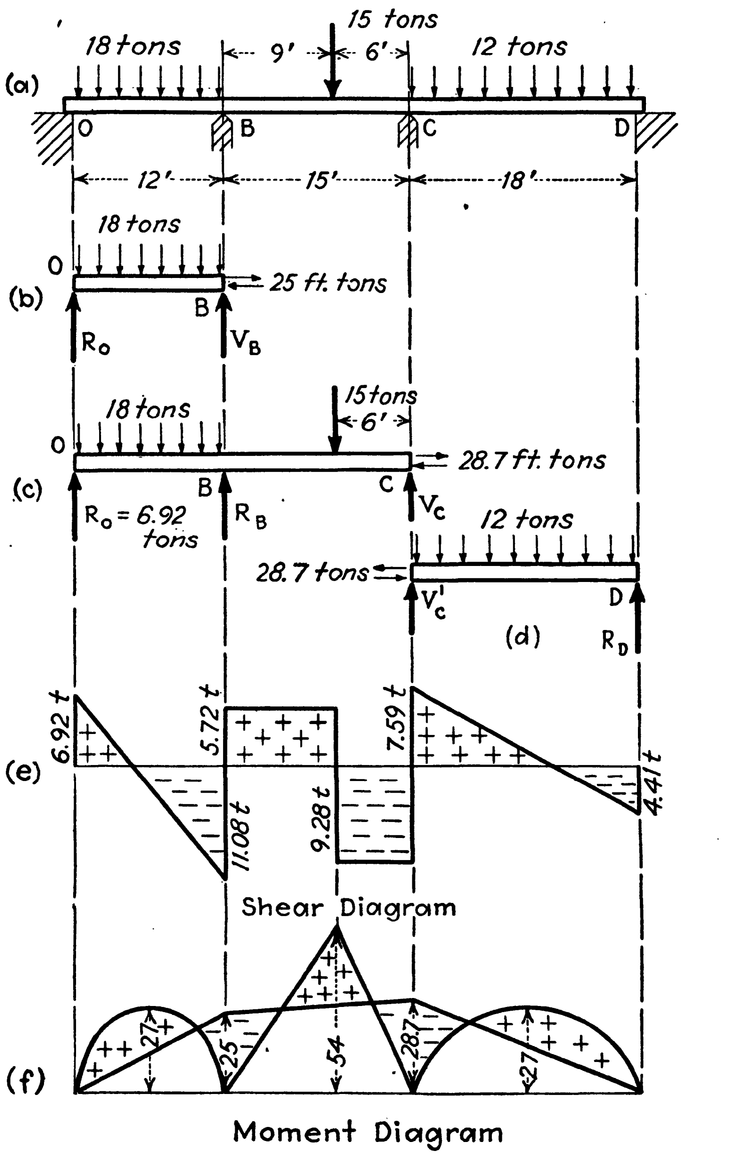

15 tons
18 tons
9'
6'
12 tons
(a)
O
B
C
D
12'
15'
18'
18 tons
O
25 ft. tons
(b)
B
R_o
V_B
15 tons
18 tons
6'
O
28.7 ft. tons
(c)
B
C
R_o = 6.92 tons
R_B
V_c
12 tons
28.7 tons
V_c'
D
(d)
R_D
6.92 t
5.72 t
7.59 t
(e)
11.08 t
9.28 t
4.41 t
Shear Diagram
27
25
54
28.7
27
(f)
Moment Diagram

FIG. 338.

Applying the theorem of three moments to OBC,

$$0 + \frac{M_B(12 + 15)}{3} + \frac{M_C \times 15}{6} + \frac{18(12)^2}{24} + \frac{15 \times 6 \times 9 \times 21}{6 \times 15} = 0$$

Or, simplified,

$$18M_B + 5M_C + 594 = 0. \qquad (a)$$

Applying the theorem to BCD,

$$\frac{M_B \times 15}{6} + \frac{M_C(15 + 18)}{3} + 0 + \frac{15 \times 9 \times 6 \times 24}{6 \times 15} + \frac{12(18)^2}{24} = 0.$$

Or, simplified,

$$5M_B + 22M_C + 756 = 0. \qquad (b)$$

Solving Eqs. (*a*) and (*b*),

$$M_B = -25 \text{ ft.-tons}, \qquad \text{and} \qquad M_C = -28.7 \text{ ft.-tons}.$$

To find the reaction at O, take OB free (Fig. 338*b*), represent the stress couple at B as shown, and write the summation of moments about B equal to zero. Then $R_O = 6.92$ tons.

Next, take OBC free (Fig. 338*c*), and write Σ moments about $C = 0$;

$$6.92 \times 27 + R_B \times 15 - 18 \times 21 - 15 \times 6 + 28.7 = 0.$$

Therefore

$$R_B = 16.8 \text{ tons}.$$

With CD free (Fig. 338*d*), $R_D = 4.41$ tons. Finally, Σ vertical forces $= 0$ gives $R_C = 16.87$ tons.

208. Beams with Supports Not at the Same Level.—In the foregoing applications of the theorem of three moments, it was assumed that the supports of the beam were all at the same level. Accordingly, the quantity d in the three-moment equation was zero. Even if the supports were originally at the same level, they may later be out of level owing to the unequal settling of the supports. Sometimes it is important to know the effect of any given or assumed settling of the supports upon the stresses in the beam.

In considering the theorem of three moments as developed in Art. 203, it should be noted that d was taken as a downward deflection of the point B from the line joining O and C. Hence, *if B deflects upward from the line OC, d is negative in Eq.* (5).

Whenever d is not zero, the quantities E and I are left in the equation and the right-hand member of the equation may have a large value even for a small value of d.

Example. Fig. 339.—Originally, O, B, and C were at the same level. Later, owing to unequal settling of the supports, it is found that B is 0.25 in. above the line joining O and C. That is, $d = -0.25$ in. $= -\frac{1}{48}$ ft.

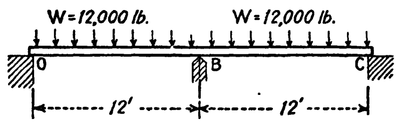


FIG. 339.

The beam is a 10-in. standard I-beam weighing 30 lb./ft. for which $I = 133.5$ in.^{4}, or $133.5/12^4$ if expressed as ft^4.

$$E = 30{,}000{,}000 \text{ lb./sq. in.} = 30{,}000{,}000 \times (12)^2 \text{ lb./sq. ft.}$$

Note.—Great care must be taken to have the units consistent. Do not use two different units for force or for distance in an equation. If desired, the inch and the pound may be used as units.*

In the foot-pound system, $E = 30{,}000{,}000 \times 12^2$; $I = \dfrac{133.5}{12^4}$; $d = -\dfrac{1}{48}$. Writing the equation of three moments,

$$0 + \frac{M_B 24}{3} + 0 + \frac{12{,}000 \times (12)^2}{24} + \frac{12{,}000 \times (12)^2}{24} =$$
$$(30{,}000{,}000 \times 12^2) \times \left(\frac{133.5}{12^4}\right) \times \left(-\frac{1}{48}\right) \times \left(\frac{1}{12} + \frac{1}{12}\right)$$

from which M_B (the moment at B) equals $-30{,}100$ ft.-lb. If the supports were at the same level (Problem 184), the moment at B would be $-WL/8 = -18{,}000$ ft.-lb.

From these values it is seen that in this example the moment at B is increased more than 65 per cent if the support at B is 0.25 in. above the line OC.

209. Deflection of a Point in a Beam.—In the theorem of three moments, O, B, and C are three points in the beam (not necessarily points of support) and d is the deflection of the point B

* The solution of this problem with the inch and the pound as units makes a good exercise in the use of units.

from the line joining O and C (Fig. 317, Art. 203). This theorem, then, may be used to determine d in terms of E, I, and other constants, provided the moments at O, B, and C are known, *i.e.*, provided M_O, M_B, and M_C are known. *If the value of d thus found is positive, the deflection of the point B from the line OC is downward; if negative, the deflection is upward.* Note that OB of length L_1 is the left panel, and that BC of length L_2 is the right panel.

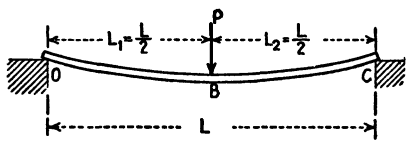


FIG. 340.

Example I. Fig. 340.—A prismatic homogeneous beam rests on two end supports (a simple beam). It is loaded with a single load P at the center B. Find the deflection of B from the line joining O and C. $M_O = 0$, and $M_C = 0$; $M_B = +PL/4$. Since the panels OB and BC are not loaded (Note 7, Art. 203),

$$\frac{A_1\bar{x}_1}{L_1} = 0, \qquad \text{and} \qquad \frac{A_2\bar{x}_2}{L_2} = 0.$$

The three moment equation for the points O, B, and C becomes, therefore,

$$0 + \frac{\frac{PL}{4} \times (L_1 + L_2)}{3} + 0 + 0 + 0 = dEI\left(\frac{1}{L_1} + \frac{1}{L_2}\right).$$

Or, since $L_1 = L_2 = L/2$,

$$\frac{PL^2}{12} = \frac{4dEI}{L}.$$

Therefore

$$d = \frac{PL^3}{48EI}.$$

Example II. Fig. 341.—Same as Example I except that the

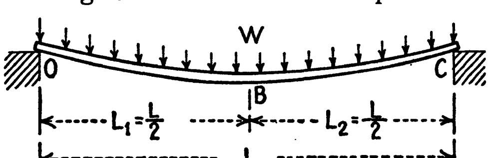


FIG. 341.

beam is loaded uniformly over the entire span.

$$M_O = 0, \qquad M_C = 0, \qquad M_B = \frac{WL}{8}.$$

Also (Case V, Art. 204),

$$\frac{A_1\bar{x}_1}{L_1} = \frac{A_2\bar{x}_2}{L_2} = \frac{\frac{W}{2} \times \left(\frac{L}{2}\right)^2}{24} = \frac{WL^2}{192}.$$

Therefore

$$0 + \frac{\frac{WL}{8} \times L}{3} + 0 + \frac{2WL^2}{192} = dEI\left(\frac{1}{L_1} + \frac{1}{L_2}\right) = \frac{4dEI}{L};$$

$$d = \frac{5}{384}\frac{WL^3}{EI}.$$

210. Beam Built In at One End and Resting on a Support at the Other End. Supports at the Same Level. Fig. 342.—For convenience, a concentrated load P is used. The results, however, are applicable to any system of loading. Replace the beam by a continuous beam on three supports, symmetrical and symmetrically loaded with respect to B, and apply the theorem of three moments to the points O, B, and O' (Fig. 342b).

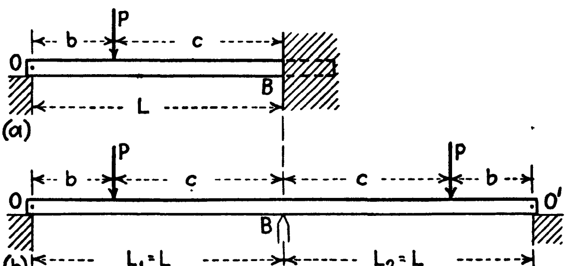


FIG. 342.

$$M_O = 0, \qquad M_{O'} = 0, \qquad d = 0, \qquad L_1 = L_2 = L.$$

$$\frac{A_1\bar{x}_1}{L_1} = \frac{A_2\bar{x}_2}{L_2} = \frac{A\bar{x}}{L}\bigg]_O^B.$$

$$0 + \frac{M_B \times 2L}{3} + 0 + \frac{2A\bar{x}}{L}\bigg]_O^B = 0.$$

Or, letting $H_{OB} = \frac{A\bar{x}}{L}\bigg]_O^B$ and solving for M_B,

$$M_B = -\frac{3H_{OB}}{L}. \tag{6}$$

Example I.—Assume the beam loaded as shown in Fig. 342a.

$$H_{OB} = \frac{Pbc(L + b)}{6L} \text{ (Case I, Art. 204).}$$

$$M_B = -\frac{Pbc(L + b)}{2L^2}.$$

If P is *centrally* applied, $b = c = L/2$

$$M_B = -\tfrac{3}{16}PL \text{ (see Problem 183).}$$

Example II.—Assume the beam uniformly loaded, $W = wL$.

$$H_{OB} = \tfrac{1}{24}WL^2 \text{ (Case V, Art. 204).}$$
$$M_B = -\tfrac{1}{8}WL.$$

Problem 189.—Assume the beam triangularly loaded (Fig. 343). Find M_B.

Ans. $M_B = -\tfrac{2}{15}WL.$

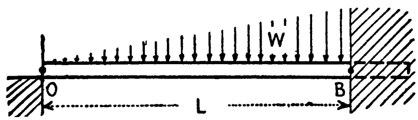


FIG. 343.

STATICALLY INDETERMINATE FRAMES

211. Modern steel-frame or reinforced-concrete buildings are *statically indeterminate* often to a high order (Art. 111). In the five-story building of Fig. 344, for instance, the beams and columns are riveted or welded together in the case of a steel building or are built together integrally (at the joints) in the case of a concrete building. Assume the floors loaded as shown. Such a loading is possible (although not probable) and the stresses in the various members of the building are determined on the assumption that the floors may be loaded in this critical way. Moreover, the building must be able to withstand a heavy lateral wind pressure, as shown in Fig. 344.

If our attention is directed to the panel OB (any panel in the frame), we should note that the joint O will rotate since OB, OC, OD, and OE (as well as all other members of the structure) will deform slightly. In like manner the joint B will rotate. That is, the beam OB is not fully restrained at O and B. To determine the stresses induced in the various members of the frame it is necessary, therefore, to consider the rotation of the joints. Theoretically, the rotation of a joint such as O will depend upon the

length and rigidity of all the individual members of the frame. In design, the process is greatly simplified, however, by making certain assumptions that for practical purposes will give sufficiently close approximations to the stresses.

In the solution of statically indeterminate frames, the *slope deflection method* may be used to great advantage. In the remaining articles of this chapter two equations will be developed—equations that in one form or another are used almost universally at present. In these equations O and B will designate, respectively, the left and the right end of any given panel in the frame.

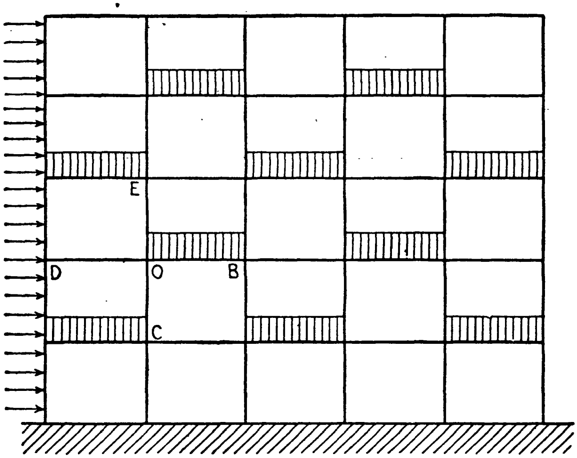


FIG. 344.

Note.—The moment area will be used and it is very important to see clearly what is meant by the expressions

$$H_{OB} = \frac{A\bar{x}}{L}\Big]_O^B \quad \text{and} \quad H_{BO} = \frac{A\bar{x}}{L}\Big]_B^O.$$

In these expressions, A equals the area of normal moment diagram under OB (Art. 201). That is, A is the area of the moment diagram that would be obtained if OB were a simple beam on end supports. In H_{OB}, $\bar{x}$ is the distance of the centroid of A from the point O. In H_{BO}, $\bar{x}$ is the distance of the centroid of A from the point B.

212. Beam with Both Ends Built In. Supports at the Same Level. Tangents at Built-in Ends Remain Horizontal. Fig. 345.—Required to find M_O and M_B.

The deflection of O from B's tangent equals zero. That is, $\Delta y\Big]_O^B = 0$. Also the deflection of B from O's tangent equals zero. That is, $\Delta y\Big]_O^B = 0$. The solution of the two equations will determine M_O and M_B.

To show how the theorem of three moments may be used to solve problems of this kind, conceive the beam replaced by a continuous beam on four supports (Fig. 345*b*). Note that $B'O$

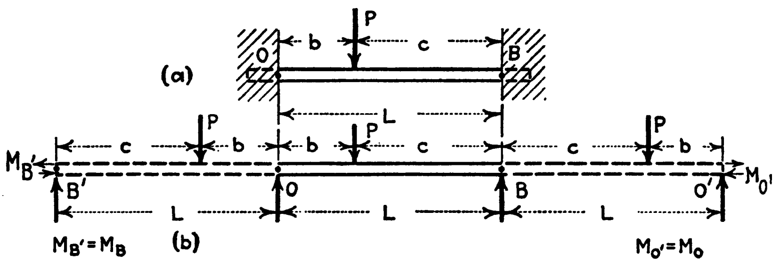


FIG. 345.

and OB must be symmetrical and symmetrically loaded with respect to O to insure that the tangent at O remains horizontal. Hence $M_{B'} = M_B$. In like manner OB and BO' must be symmetrical and symmetrically loaded with respect to B. Hence $M_{O'} = M_O$. Apply the theorem of three moments to $B'OB$, remembering that $M_{B'} = M_B$ and $H_{B'O} = H_{BO}$ (Art. 211, note).

$$\frac{M_BL}{6} + \frac{M_O2L}{3} + \frac{M_BL}{6} + 2H_{BO} = 0.$$

Apply the theorem of three moments to OBO', remembering that $M_{O'} = M_O$ and $H_{OB} = H_{O'B}$.

$$\frac{M_OL}{6} + \frac{M_B2L}{3} + \frac{M_OL}{6} + 2H_{OB} = 0.$$

These equations when simplified become

$$\frac{M_BL}{3} + \frac{2M_OL}{3} + 2H_{BO} = 0, \qquad \frac{M_OL}{3} + \frac{2M_BL}{3} + 2H_{OB} = 0.$$

The solution of the last two equations gives

$$\left.\begin{aligned} M_O &= -\frac{2}{L}(2H_{BO} - H_{OB}). \\ M_B &= -\frac{2}{L}(2H_{OB} - H_{BO}). \end{aligned}\right\} \qquad (7)$$

Example I.—Beam carries a single concentrated load (Fig. 345*a*).

$$H_{OB} = \frac{Pbc(L+b)}{6L} \text{ (Case I, Art. 204).}$$

$$H_{BO} = \frac{Pbc(L+c)}{6L}.$$

$$2H_{BO} - H_{OB} = \frac{Pbc}{6L}(2L + 2c - L - b) = \frac{Pbc}{6L}(L + 2c - b) =$$

$$\frac{Pbc}{6L}(b + c + 2c - b) = \frac{Pbc^2}{2L}.$$

Hence [Eqs. (7)]

$$M_O = -\frac{Pbc^2}{L^2}. \tag{8a}$$

In like manner

$$M_B = -\frac{Pb^2c}{L^2}. \tag{8b}$$

Example II.—Beam carries a uniformly distributed load over its whole length.

$$H_{OB} = H_{BO} = \frac{WL^2}{24} \text{ (Case V, Art. 204).}$$

Hence

$$\left.\begin{aligned} M_O &= -\tfrac{1}{12}WL. \\ M_B &= -\tfrac{1}{12}WL. \end{aligned}\right\} \tag{9}$$

213. Beam Carries No Load. Supports Not on the Same Level. End Tangents Not Horizontal. Fig. 346*a*.—Required to find M_O and M_B. Figure 346*b* gives the complete moment diagram for OB (no load on beam). By using the moment area, the deflection of O from B's tangent is [Eq. (4), Art. 200]

$$\begin{aligned} \Delta y\Big]_O^B = d - L\theta_B &= \frac{1}{EI}\left(\frac{M_OL}{2} \times \frac{L}{3} + \frac{M_BL}{2} \times \frac{2}{3}L\right) \\ &= \frac{L^2}{6EI}(M_O + 2M_B). \end{aligned} \tag{a}$$

Or, dividing through by L and letting $d/L = R = \angle O'BO$ in radians (deflection small), Eq. (*a*) becomes

$$R - \theta_B = \frac{L}{6EI}(M_O + 2M_B). \tag{b}$$

The angle between the tangent at O and the tangent at B is [Eq. (3), Art. 200]

$$\Delta\phi\Big]_O^B = \theta_O - \theta_B = \frac{1}{EI}\left(\frac{M_O L}{2} + \frac{M_B L}{2}\right) = \frac{L}{2EI}(M_O + M_B). \quad (c)$$

Solving Eqs. (b) and (c) for M_O and M_B,

$$\left.\begin{aligned} M_O &= -\frac{2EI}{L}(3R - 2\theta_O - \theta_B). \\ M_B &= \frac{2EI}{L}(3R - 2\theta_B - \theta_O). \end{aligned}\right\} \quad (10)$$

Note.—It is necessary to be consistent in sign. In Fig. 346*a*, all quantities are taken as positive. That is, M_O or M_B is positive

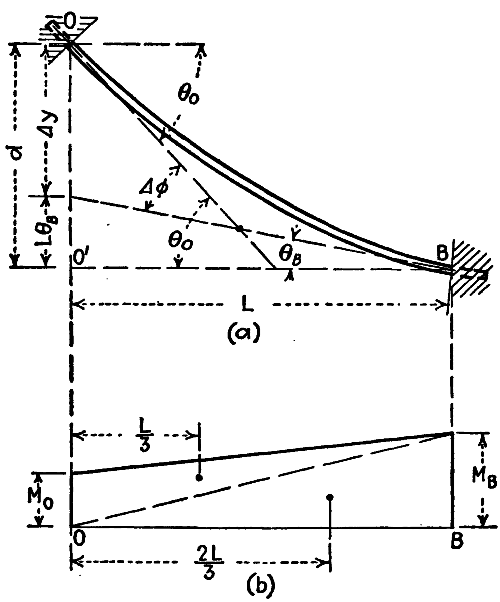


FIG. 346.

(+) if the beam bends so as to produce *compression in the upper fibers;* θ_O or θ_B is positive if the tangent at O or B *rotates clockwise.* R is positive if the straight line OB *rotates clockwise.*

214. General Case.—By the law of superposition the results of Art. 212 [Eq. (7)] or, for special cases, Eq. (8) or (9) may be combined with the results of Art. 213 [Eqs. (10)] with the precaution that the signs must be consistent.

Example. Fig. 347.—A 12-in. 40.8-lb. standard I-beam, 15 ft. long, is built in at both ends and carries a uniformly distributed load of $W = 50{,}000$ lb. Originally O's tangent and B's tangent coincided but later, owing to the settling of the pier B, B was $\frac{1}{4}$ in. below the level of O, and B's tangent had rotated clockwise through $\frac{1}{4}°$ (0.00393 radian).

Find

1. M_O and M_B, before the pier settled.
2. M_O and M_B due to settling of pier.
3. Resulting moments.

1. Using Eq. (9), Art. 212,

$$M_O = M_B = -\frac{WL}{12} = -\frac{50{,}000 \times 15 \times 12}{12} = -750{,}000 \text{ in.-lb.}$$

2. Use Eqs. (10), Art. 213, to find M_O and M_B due to settling of pier (no load on beam).

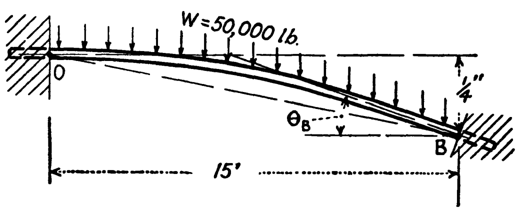


FIG. 347.

Take $E = 30{,}000{,}000$ lb./sq. in. From a steel company's handbook, $I = 269$ in.^{4}

$d = \frac{1}{4}$ in. $L = 15$ ft. $= 180$ in. Therefore

$$R = d/L = +0.00139 \quad \text{radian.}$$

Note that R is plus since the line OB rotates clockwise.

$$\theta_O = 0, \qquad \theta_B = +0.00393 \text{ radian.}$$

Note that θ_B is plus since tangent at B rotates clockwise.

Therefore [Eqs. (10)], M_O due to settling of pier is

$$M_O = \frac{-2EI}{L}(3R - 2\theta_O - \theta_B) = \frac{-2 \times 30{,}000{,}000}{180} \times$$
$$269(3 \times 0.00139 - 0 - 0.00393) = -21{,}500 \text{ in.-lb.}$$

In like manner,

$$M_B = \frac{2 \times 30{,}000{,}000}{180} \times 269(3 \times 0.00139 - 2 \times 0.00393 - 0) = -331{,}000 \text{ in.-lb.}$$

3. By adding algebraically, the total moments at O and B are, respectively,

$$M_O = -750{,}000 - 21{,}500 = -771{,}500 \text{ in.-lb.}$$
$$M_B = -750{,}000 - 331{,}000 = -1{,}081{,}000 \text{ in.-lb.}$$

Note.—With 18,000 lb./sq. in., the safe moment in the beam is $M_m = 807{,}000$ in.-lb. Hence, after the pier has settled, the fiber stress is excessive.

PROBLEMS

190. A prismatic homogeneous cantilever carries an end load P (Fig. 348). Find the deflection of the mid-point O.

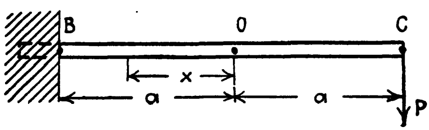


FIG. 348.

Suggestion.—Take the point O as an origin and integrate between O and B. Use the slope deflection method [Eq. (2), Art. 197].

Ans. $\Delta y = \frac{5}{6}Pa^3/EI$.

191. In Problem 190, move the load to O and find the deflection of C. Note now that C is the origin. Compare result with that of Problem 190.

192. A beam of length L rests on end supports and carries a central concentrated load P. Find the deflection of the quarter points. Find separately the deflection of the end and of the quarter point, from the tangent drawn at the middle of the beam. *Ans.* $\frac{11}{768}PL^3/EI$.

193. In Problem 190, use the moment area method.

194. In Problem 191, use the moment area method.

195. A 10-in. 30-lb. standard steel I-beam, 10 ft. long, has one end B built in horizontally, the other end O resting on a support. The beam carries a central concentrated load of 20,000 lb. Assume that the support O sinks $\frac{1}{4}$ in. below the level of B. Find the reaction of the support at O and construct the shear and moment diagrams (Note 4, Art. 198; Ex. V, Art. 199). *Ans.* $R_O = 4512$ lb.

196. In Problem 195, assume that the support O moves up $\frac{1}{4}$ in.

197. Let OB of length $L = 16$ ft. be part of a longer beam; $M_O = 3$ ft.-tons, $M_B = -5$ ft.-tons, $P = 4$ tons placed 6 ft. from O. Construct the complete moment diagram for OB and its loading. *Ans.* $M_m = 15$ ft.-tons.

198. With reference to Fig. 330, let $L_1 = 20$ ft., $L_2 = 16$ ft., $W = 7.5$ tons, $P = 9$ tons placed 7 ft. from the end support C. Neglect the weight of the beam and find the economical standard steel I-beam. $S = 18{,}000$ lb./sq. in.

Ans. $M_m = 25.9$ ft.-tons.

199. In Problem 198, consider the weight of the beam.

200. A beam rests on three supports at the same level (Fig. 349). Apply the theorem of three moments to the points O, B, and C and find M_B.

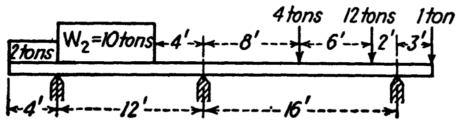


FIG. 349.

What is the sign of M_O and M_C? Construct the shear and the moment diagram. *Ans.* $M_B = -18.56$ ft.-tons; $R_C = 12.53$ tons.

201. In Problem 200, select an economical I-beam. Take $S = 18{,}000$ lb./sq. in. Neglect the weight of the beam.

202. In Problem 201, consider the weight of the beam.

203. A beam rests on four supports at the same level. The length of each span is $L = 12$ ft. Each of the outer spans carries a central load of 5 tons and the middle span carries a uniformly distributed load of 9 tons. Find the reactions and construct the shear and the moment diagram.

Ans. $M_B = M_C = -9.9$ ft.-tons; $M_m = 10.05$ ft.-tons.

204. The cantilever of Fig. 350 is reinforced as shown. The I for the part

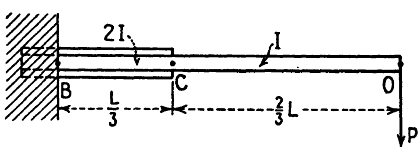


FIG. 350.

CB is twice the I for the part CO. Find the deflection of O. Neglect the weight of the beam.

Suggestion: $\int_O^B = \int_O^C + \int_C^B$

Ans. $\frac{35}{162}PL^3/EI$.

205. A 12-in. steel I-beam, AD, 22 ft. long, is strengthened with flange

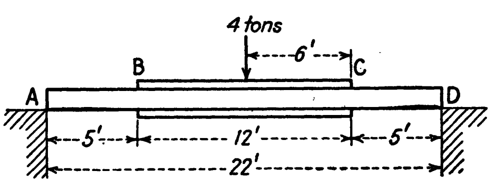


FIG. 351.

plates between points B and C (Fig. 351) and is simply supported at its two extremities (at same level).

The moment of inertia of the cross-section anywhere between A and B and also between C and D is 200 in.^{4}, while for any section between B and C it is 300 in.^{4} A load of 4 tons is placed in the middle of the span. From B to C is 12 ft.

Compute the deflection of the middle point of the beam, neglecting its weight, and assuming the modulus of eleasticity E as 30,000,000 lb./sq. in.

Ans. 0.357 in.

CHAPTER X

STRESS INTENSITIES ON DIFFERENT PLANES THROUGH A POINT IN A BODY

215. Introduction.—In the foregoing chapters, formulas were derived which in many cases enable one to calculate the intensities of stress at a point in a body on each of two mutually perpendicular planes. For instance (Fig. 352), at a point E in a beam the intensity of the normal stress on a vertical plane CD (right section) may be found by means of the formula (Art. 162)

$$S = \frac{My}{I}$$

and the intensity of the shearing stress at E on the horizontal

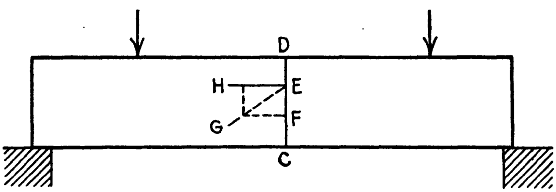


FIG. 352.

plane EH (and therefore also on the vertical plane EC) may be found by means of the formula [Eq. (11), Art. 163]

$$S_s = \frac{VA\bar{y}}{tI}.$$

Moreover, the stress on the vertical plane HEC (plane parallel to the plane of the paper) is zero or may be assumed to be zero. Hence at a point in a beam the intensity of stress on each of three mutually perpendicular planes is known or may be found.

Let the intensities of stress at a point in a body on three mutually perpendicular planes be known. The question now arises whether these stresses are the most significant stresses at that point. That is, may not the intensity of a stress at the point E be greater on some oblique plane (such as EG) than on any of the three given planes?

To answer this question, consider an elemental prism at E. Figure 353 shows this prism in perspective. In Art. 18, it was shown that the stresses on a face of an elemental prism may be assumed as uniformly distributed and that the stresses on opposite faces may be assumed to be equal and opposite. Hence, by *knowing the intensities of stress on the three adjacent faces EF, EH,* and *EG* (three mutually perpendicular planes), *those on all six faces are known.*

In this chapter, the stresses on the face EG (the face parallel to the plane of the paper) are assumed equal to zero. That is, it is assumed that the stresses on one of the three mutually perpendicular planes are zero. This is the case that commonly occurs in engineering.

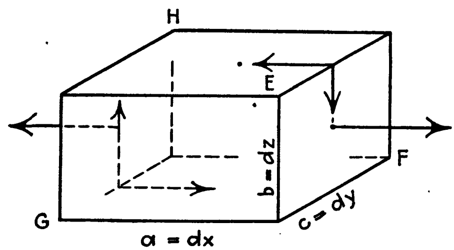


FIG. 353.

For convenience, the prism will be represented by a side view with the axes respectively horizontal and vertical (Fig. 354). Note that the dimensions a and b (c is perpendicular to the paper) are of infinitesimal magnitude. By varying the ratio of a to b, the diagonal EG may make any desired angle with the horizontal or with the vertical. The problem of finding the intensities of stress on an oblique plane through E reduces itself, therefore, to this: *If the intensities of stress on the faces of an elemental prism are known, it is required to find the intensities of stress on the diagonal planes of this prism.*

Note.—In the limit, a and b approach zero. Hence (Fig. 354) the diagonal plane FH may be thought of as an oblique plane through E.

216. Stresses Resulting from Simple Shear.—Consider first the simple case of an elemental prism upon four of whose faces *only* shearing stresses act (Fig. 355a). It is assumed that no stresses act on the two faces that are parallel to the plane of the paper. By Art. 19, the shearing stresses on the four faces are all of the

same intensity S_s. Hence the total shearing force on each of the horizontal planes is S_sac (c being the dimension of the prism $\perp$ the plane of the paper), and on each of the vertical planes it is S_sbc. Divide the prism into two halves by the diagonal plane GE and take the upper half free (Fig. 355*b*). If d is the length of the diagonal plane and S_s' is the intensity of shear on this plane, then the total shearing force on the diagonal plane is $S_s'dc$. In like manner, the total normal force on the diagonal plane is $S'dc$.

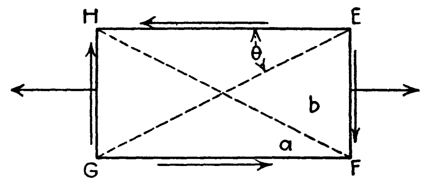


FIG. 354.

Shearing Stress on Diagonal Plane.—Take the X-axis along the diagonal plane and the Y-axis perpendicular to it as is indicated in Fig. 355*b*. Since equilibrium exists, $\Sigma F_x = 0$; *i.e.*, the summation of forces parallel to the diagonal plane must equal zero.

$$S_s'dc - (S_sac)\cos\theta + (S_sbc)\sin\theta = 0.$$

Or

$$S_s' = S_s\frac{a}{d}\cos\theta - S_s\frac{b}{d}\sin\theta.$$

From the figure, $a/d = \cos\theta$, and $b/d = \sin\theta$. Therefore

$$\begin{aligned} S_s' &= S_s\cos^2\theta - S_s\sin^2\theta = S_s(\cos^2\theta - \sin^2\theta) \\ &= S_s\cos 2\theta. \end{aligned}$$

The value of S_s' becomes a maximum numerically when $\cos 2\theta = 1$ or -1, *i.e.*, when $\theta = 0$ or $90°$. Hence, numerically, the shearing stress is a maximum on the planes EH and EF. Therefore

$$\text{Max. } S_s' = S_s. \qquad (a)$$

Normal Stress on a Diagonal Plane.—To determine the normal stress on the diagonal plane EG (Fig. 355*b*), put $\Sigma F_y = 0$.

$$S'dc - (S_sbc)\cos\theta - (S_sac)\sin\theta = 0.$$

Or

$$\begin{aligned} S' &= S_s\frac{b}{d}\cos\theta + S_s\frac{a}{d}\sin\theta \\ &= S_s\sin\theta\cos\theta + S_s\cos\theta\sin\theta = 2S_s\sin\theta\cos\theta \\ &= S_s\sin 2\theta \text{ (in simplest form).} \qquad (b) \end{aligned}$$

The value of S' as given by Eq. (*b*) becomes a maximum when $\sin 2\theta = 1$, *i.e.*, when $\theta = 45°$. Putting $\theta = 45°$ in Eq. (*b*), we obtain

$$\text{Max. } S' = S_s. \qquad (c)$$

Note.—With the shearing forces acting as shown in Fig. 355*b*, S' is a tensile stress and may be designated by S'_t. If the direction of the shearing forces are reversed, or if the other diagonal, *HF* (Fig. 355*a*), is chosen, S' becomes a compressive stress and may be

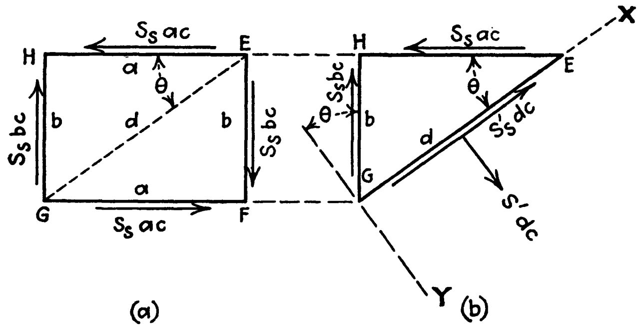


FIG. 355.

designated by S'_c. Equation (*c*) is to be interpreted to mean that *numerically*

$$\text{Max. } S'_t = \text{max. } S'_c = S_s.$$

The results of this article may be stated as follows: *If the stress at a point on each of two planes at right angles to each other is one of simple shear of intensity* S_s, then

Theorem I.—The intensity of this stress is numerically greater than that on any oblique plane through that point. That is [Eq. (*a*)],

$$\text{Max. } S'_s = S_s. \qquad (1)$$

Theorem II.—The intensity of the normal stress is numerically a maximum on planes making an angle of 45° with either of the given planes and its value is [Eq. (*c*)]

$$\text{Max. } S' = S_s. \qquad (2)$$

Illustration I.—In Chap. V, it was seen that the stress on a right section of a shaft under *pure* torsion is one of pure shear (no normal stress). It follows then from Theorem I (above) that, if a shaft is put under pure torsion, the maximum intensity

of shear occurs on a right section and hence may be determined by means of the torsion formula

$$T = \frac{S_s J}{r} \text{ (Art. 95).}$$

Illustration II. Fig. 356.—Since the stress at a point on a right section of a shaft under pure torsion is one of simple shear of intensity S_s (no normal stress), it follows (Theorem II) that max. $S' = S_s$ and that this stress occurs on oblique planes making an angle of 45° with the axis of the shaft. In the case of chalk, cast iron, and other brittle materials, the strength in tension is less than the strength in shear. Hence, if a cylinder of such material is subjected to an increasing torque until it fails, *failure will be due to tension* and will take place along a surface making approximately an angle of 45° with the axis of the cylinder.*

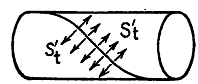


FIG. 356.

217. Stresses Resulting from Simple Normal Stresses.—Figure 357 represents an elementary prism acted upon by two sets of simple normal stresses of intensities S_1 and S_2, respectively. That is, the prism is acted upon by simple biaxial stresses.† Take free the upper half (Fig. 357*b*).

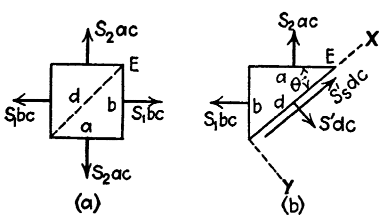


FIG. 357.

Shearing Stress Resulting from Biaxial Stresses.—Putting $\Sigma F_x = 0$,

$$S'_s dc - (S_1 bc) \cos \theta + (S_2 ac) \sin \theta = 0.$$

$$S'_s = S_1 \frac{b}{d} \cos \theta - S_2 \frac{a}{d} \sin \theta$$

$$= S_1 \sin \theta \cos \theta - S_2 \cos \theta \sin \theta.$$

Therefore

$$S'_s = \frac{S_1 - S_2}{2} \sin 2\theta. \qquad (a)$$

Evidently S'_s is numerically a maximum when $\sin 2\theta = 1$, *i.e.*, when $\theta = 45°$. Hence, numerically,

$$\text{Max. } S'_s = \frac{S_1 - S_2}{2}. \qquad (b)$$

* Take a cylindrical piece of chalk and twist it. Note how it fails.

† "Biaxial" means along two axes.

Normal Stress Resulting from Biaxial Stresses.—Putting $\Sigma F_y = 0$,

$$S'dc - (S_1bc)\sin\theta - (S_2ac)\cos\theta = 0.$$

$$S' = S_1\frac{b}{d}\sin\theta + S_2\frac{a}{d}\cos\theta.$$

Therefore

$$S' = S_1\sin^2\theta + S_2\cos^2\theta. \tag{c}$$

To find that value of θ which will make S' a maximum or a minimum, put $dS'/d\theta = 0$.

$$\frac{dS'}{d\theta} = 2S_1\sin\theta\cos\theta - 2S_2\cos\theta\sin\theta$$
$$= 2(S_1 - S_2)\sin\theta\cos\theta = 0.$$

That is, $dS'/d\theta = 0$ if $\sin\theta = 0$, or if $\cos\theta = 0$. If $\sin\theta = 0$, $\theta = 0$. If $\cos\theta = 0$, $\theta = 90°$.

$$\left.\begin{aligned}&\text{Putting } \theta = 0 \text{ in Eq. } (c), && S' = S_2.\\ &\text{Putting } \theta = 90°, && S' = S_1.\end{aligned}\right\} \tag{d}$$

Hence, *if the stresses at a point on two mutually perpendicular planes are simple normal stresses of intensities* S_1 and S_2, *respectively*, then

Theorem III.—One stress is the normal stress of maximum intensity, and the other is the normal stress of minimum intensity at that point. That is [Eqs. (*d*)],

$$\text{Max. } S' = S_1 \text{ or } S_2; \qquad \text{min. } S' = S_2 \text{ or } S_1. \tag{3}$$

Note.—In Fig. 357 the stresses S_1 and S_2 are represented as tensile (+). One or both, however, may be compressive (−). Assume now that as θ varies in Eq. (*c*) S' varies continuously from $S_1 = 4000$ to $S_2 = -8000$. In the calculus sense, $S_1 = 4000$ is the maximum value of S' and $S_2 = -8000$ is the minimum value of S'. By our convention of signs a tensile stress is designated as positive (+) and a compressive stress as negative (−) and it is convenient to say that $S_1 = 4000$ is the maximum *tensile* stress and $S_2 = 8000$ is the maximum *compressive* stress.

Theorem IV.—The intensity of the shearing stress is numerically a maximum on a plane making an angle of 45° with either of the given planes and its value is [Eq. (*b*)]

$$\text{Max. } S_s' = \frac{S_1 - S_2}{2}. \tag{4}$$

Example I.—A boiler is subjected to an internal steam pressure. Let S_2 equal the unit tensile stress on a longitudinal section and S_1 equal that on the circumferential section of the shell (see Fig. 358). In Art. 72, it was shown that $S_2 = 2S_1$. Since S_2 and S_1 are simple normal stresses, S_2 is the maximum and S_1 is the minimum unit tensile stress induced in the shell (Theorem III).

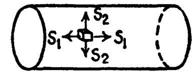


FIG. 358.

If $S_2 = 16{,}000$ lb./sq. in., then $S_1 = 8000$ lb./sq. in. Therefore (Theorem IV),

$$\text{Max. } S'_s = \frac{S_1 - S_2}{2} = -4000 \text{ lb./sq. in.}$$

That is, numerically,

$$\text{Max. } S'_s = 4000 \text{ lb./sq. in.}$$

If 16,000 lb./sq. in. is taken as the allowable tensile stress, then 10,000 lb./sq. in. may be taken as the allowable shearing stress. Hence, in a boiler, shearing stresses in the shell need not be considered.

Example II. Fig. 359.—Let $S_1 = 18{,}000$ lb./sq. in. (tension).

$$S_2 = -16{,}000 \text{ lb./sq. in. (compression).}$$

$$\text{Max. } S'_s = \frac{18{,}000 + 16{,}000}{2} = 17{,}000 \text{ lb./sq. in.}$$

If 12,000 lb./sq. in. is considered the maximum safe unit shear, then 17,000 lb./sq. in. is excessive.

From Theorem III, max. $S' = S_1 = 18{,}000$ lb./sq. in. (tension).

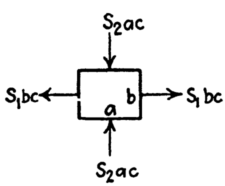


FIG. 359.

Problem 206.—A prism is under simple tension. Let P equal the pull in the prism and A equal the area of right section. Make use of the theorems and show that (1) max. $S' = P/A$; (2) max. $S'_s = P/2A$.

218. Stresses Resulting from Shearing and Axial Stresses.—Assume that shearing stresses act on four faces of an elemental prism and that normal stresses act on one pair of opposite faces (Fig. 360). That is, assume that the prism is acted upon by shearing and axial stresses.

Putting $\Sigma F_x = 0$ (Fig. 360b),

$$S'_s dc - (Sbc) \cos \theta - (S_s ac) \cos \theta + (S_s bc) \sin \theta = 0.$$

Or

$$S_s' = S\frac{b}{d}\cos\theta + S_s\frac{a}{d}\cos\theta - S_s\frac{b}{d}\sin\theta.$$

$$S_s' = S\sin\theta\cos\theta + S_s\cos^2\theta - S_s\sin^2\theta.$$

Therefore

$$S_s' = \frac{S}{2}\sin 2\theta + S_s\cos 2\theta. \tag{5}$$

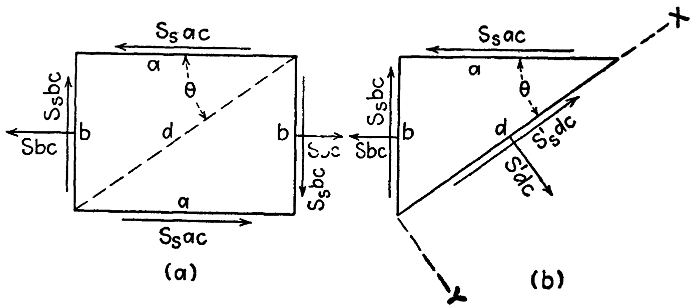


FIG. 360.

Putting $\Sigma F_y = 0$,

$$S'dc - (Sbc)\sin\theta - (S_sac)\sin\theta - (S_sbc)\cos\theta = 0.$$

Or

$$S' = S\frac{b}{d}\sin\theta + S_s\frac{a}{d}\sin\theta + S_s\frac{b}{d}\cos\theta;$$

$$S' = S\sin^2\theta + S_s\cos\theta\sin\theta + S_s\sin\theta\cos\theta;$$

$$S' = \frac{S}{2}(1 - \cos 2\theta) + S_s\sin 2\theta.$$

Therefore

$$S' = \frac{S}{2} - \frac{S}{2}\cos 2\theta + S_s\sin 2\theta. \tag{6}$$

Note.—Equations (5) and (6) were derived on the supposition that S and S_s have the directions indicated in Fig. 360. If S or S_s is reversed in direction, it should be given the negative sign when substituted in these equations.

219. Maximum Unit Shearing Stress. Fig. 360.—The unit shearing stress on a diagonal plane is [Eq. (5), Art. 218]

$$S_s' = \frac{S}{2}\sin 2\theta + S_s\cos 2\theta. \tag{a}$$

To find that value of θ which will make S'_s a maximum (or a minimum) put $dS'_s/d\theta = 0$.

$$\frac{dS'_s}{d\theta} = S \cos 2\theta - 2S_s \sin 2\theta = 0$$

Therefore

$$\tan 2\theta = \frac{\left(\frac{S}{2}\right)}{S_s}. \tag{7}$$

There are always two angles, 180° apart, whose tangents are equal. Thus $\tan \alpha = \tan (\alpha + 180)$. Accordingly, if α is the least value of 2θ that will satisfy Eq. (7), then $2\theta = \alpha$, or $2\theta = \alpha + 180$. Therefore

$$\theta = \frac{\alpha}{2} \qquad \text{or} \qquad \theta = \frac{\alpha}{2} + 90.$$

That is, there are two values of θ, 90° apart, that will satisfy Eq. (7). One value of θ will give maximum S'_s and the other will give minimum S'_s.

The maximum and the minimum value of S'_s may be found by substituting in Eq. (a) the values of $\sin 2\theta$ and $\cos 2\theta$ as determined by Eq. (7). These values of $\sin 2\theta$ and $\cos 2\theta$ may be found from trigonometric formulas.* A better way perhaps is as follows:

Construct a right triangle ABC (Fig. 361) with S_s as base and $S/2$ as altitude. For convenience assume S and S_s as positive (+). The hypotenuse of this triangle is $\sqrt{(S/2)^2 + S_s^2}$. Since $\tan 2\theta$ is positive [Eq. (7)], this triangle may be in the first quadrant or in the third quadrant.

Therefore

$$\sin 2\theta = \pm \frac{\frac{S}{2}}{\sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}} \quad \text{and} \quad \cos 2\theta = \pm \frac{S_s}{\sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}}. \tag{b}$$

Note that in the first quadrant both $\sin 2\theta$ and $\cos 2\theta$ are positive (+) and that in the third quadrant both are negative (−). Hence for the first value of 2θ use the upper signs, and for the

* $\sin 2\theta = \pm \dfrac{\tan 2\theta}{\sqrt{1 + \tan^2 2\theta}}$, $\qquad \cos 2\theta = \pm \dfrac{1}{\sqrt{1 + \tan^2 2\theta}}$.

second value of 2θ use the lower signs. Substituting the values of $\sin 2\theta$ and $\cos 2\theta$ as given by Eqs. (*b*) in Eq. (*a*),

$$S'_s = \pm\frac{\left(\frac{S}{2}\right)^2 + S_s^2}{\sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}} = \pm\sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}. \qquad (c)$$

That is, the *intensity of the shearing stress is numerically a maximum on each of two planes at right angles to each other and its value is*

$$\text{Max. } S'_s = \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}. \qquad (8)$$

Note.—In the calculus sense, the upper (plus) sign in Eq. (*c*) gives maximum S'_s and the lower (minus) sign gives minimum

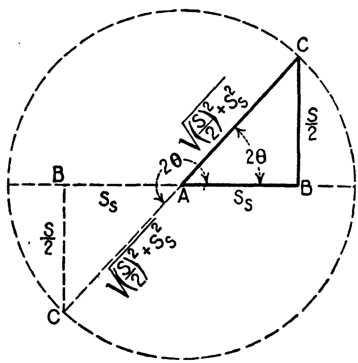


FIG. 361.

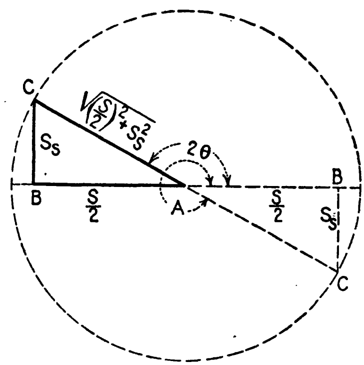


FIG. 362.

S'_s. The two values of S'_s are numerically equal as they ought to be since the shearing stresses at a point on two planes at right angles to each other always are numerically of the same intensity (Art. 19).

Example.—With reference to Fig. 360*a*, let $S = 16{,}000$ lb./sq. in. and $S_s = 10{,}000$ lb./sq. in. From Eq. (8),

$$\text{Max. } S'_s = \sqrt{8000^2 + 10{,}000^2} = 12{,}800 \text{ lb./sq. in.}$$

From Eq. (7),

$$2\theta = \tan^{-1}\frac{8000}{10{,}000} = \tan^{-1} 0.8 = 38° \, 40' \text{ or } 218° \, 40'.$$

Therefore

$$\theta = 19° \, 20' \text{ or } 109° \, 20'.$$

220. Maximum Unit Normal Stress. Fig. 360.—The normal stress on the diagonal plane is [Eq. (6), Art. 218]

$$S' = \frac{S}{2} - \frac{S}{2}\cos 2\theta + S_s \sin 2\theta. \qquad (a)$$

To find that value of θ that will make S' a *maximum or a minimum*, put $dS'/d\theta = 0$.

$$\frac{dS'}{d\theta} = S \sin 2\theta + 2S_s \cos 2\theta.$$

Therefore

$$\tan 2\theta = -\frac{S_s}{\left(\frac{S}{2}\right)}. \qquad (9)$$

If α is the least value of 2θ that will satisfy Eq. (9), then $2\theta = \alpha$ or $2\theta = \alpha + 180°$. Hence

$$\theta = \frac{\alpha}{2} \qquad \text{or} \qquad \theta = \frac{\alpha}{2} + 90°.$$

That is, the two planes are 90° apart. The normal stress is a maximum on one of these planes and a minimum on the other.

Since $\tan 2\theta$ is negative [Eq. (9)], the angle 2θ is either in the second quadrant or in the fourth. To find $\sin 2\theta$ and $\cos 2\theta$ as determined by Eq. (9), construct the right triangle ABC (Fig. 362) with S_s as the altitude and $S/2$ as the base. From this triangle we obtain

$$\left.\begin{aligned} \sin 2\theta &= \pm\frac{S_s}{\sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}}, \\ \cos 2\theta &= \mp\frac{\frac{S}{2}}{\sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}}. \end{aligned}\right\} \qquad (b)$$

Note that in the second quadrant the sine is positive (+) and the cosine is negative (−), and that in the fourth quadrant the sine is negative and the cosine is positive. Hence in Eqs. (b) the upper signs go together and the lower signs go together.

Substituting the values of $\sin 2\theta$ and $\cos 2\theta$, as determined by Eqs. (b), in Eq. (a),

$$S' = \frac{S}{2} \pm \frac{\left(\frac{S}{2}\right)^2 + S_s^2}{\sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}} = \frac{S}{2} \pm \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}.$$

Or [Eq. (8), Art. 219]

$$S' = \frac{S}{2} \pm \text{max. } S_s'. \qquad (c)$$

The normal stress S in Eq. (c) may be tensile (+) or compressive (−). Since numerically $\sqrt{(S/2)^2 + S_s^2}$ is always greater than $S/2$, the first equation (upper sign) always gives a positive value for S' (tensile stress; designate it by Max. S_t') and the second equation (lower sign) always gives a negative value for S' (compressive stress; designate it by Max. S_c'). Hence Eq. (c) may be written

$$\left.\begin{aligned}\text{Max. } S_t' &= \frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2} = \frac{S}{2} + \text{max. } S_s'.\\ \text{Max. } S_c' &= \frac{S}{2} - \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2} = \frac{S}{2} - \text{max. } S_s'.\end{aligned}\right\} \qquad (10)$$

Note.—Evidently, if S is negative (compressive stress), S_c' is numerically larger than S_t'. Frequently, however, the sign of the stress at a point is immaterial. If Max. S' designates the numerical value of the maximum normal stress at a point (tensile or compressive as the case may be), then

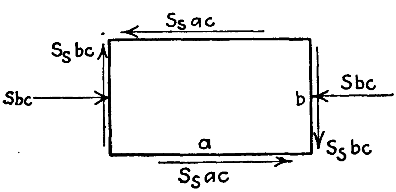


FIG. 363.

$$\text{Max. } S' = \frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2} = \frac{S}{2} + \text{max. } S_s'. \qquad (10)$$

in which the numerical value of S must be used.

Example. Fig. 363.—Let $S = -16{,}000$ lb./sq. in. and

$$S_s = 10{,}000 \text{ lb./sq. in.}$$

From Eq. (8),

$$\text{Max. } S_s' = \sqrt{8000^2 + 10{,}000^2} = 12{,}800 \text{ lb./sq. in.}$$

From Eqs. (10),

$$\text{Max. } S'_t = -8000 + 12{,}800 = 4800 \text{ lb./sq. in.}$$
$$\text{Max. } S'_c = -8000 - 12{,}800 = -20{,}800 \text{ lb./sq. in.}$$

Therefore

$$\text{Max. } S' = 20{,}800 \text{ lb./sq. in. (compression).}$$

Problem 207.—In an elemental block subject to shear and uniaxial stress,* the maximum normal stress is 12,000 lb./sq. in. and the maximum shearing stress is 9000 lb./sq. in. Find the normal and shearing stresses on the faces of this block. *Ans.* S = 6000 lb./sq. in.; S_s = 8485 lb./sq. in.

221. Stresses on an Oblique Section of a Beam. *I-beams.*—Consider a W 47-lb. beam (Fig. 364). The beam is 17.9 in.

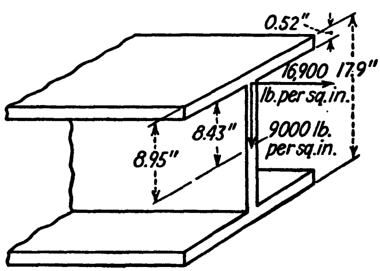


FIG. 364.

high and the flange is 0.52 in. thick. The distance of the junction of flange and web from the neutral axis is, therefore, 8.43 in.

If at a given section the allowable flexure stress

(18,000 lb./sq. in.)

is developed in the outer fibers, then the normal stress at the junction of web and flange is (Art. 135),

$$S = \frac{y}{c} \times 18{,}000 = \frac{8.43}{8.95} \times 18{,}000 = 16{,}960 \text{ lb./sq. in.}$$

Assume that at this point on the section the unit shearing stress is

$$S_s = 9000 \text{ lb./sq. in.}$$

The maximum shearing stress at this point on an oblique plane is, then,

$$\text{Max. } S'_s = \sqrt{8480^2 + 9000^2} = 12{,}350 \text{ lb./sq. in.}$$

* "Uniaxial" means along one axis.

And the maximum normal stress is

$$\text{Max. } S' = 8480 + 12{,}350 = 20{,}830 \text{ lb./sq. in.}$$

This illustration shows that, in exceptional cases, the resulting normal stress or the resulting shearing stress at some point in an I-beam may exceed the allowable value even though the direct flexural stress and the direct shearing stress in that beam do not exceed the allowable values.*

In practice, as a rule, the design of a beam is based upon the direct flexural and shearing stresses. If the resulting normal stress or the resulting shearing stress at some point in this beam should exceed or even approach the allowable stress, it is advisable at this point to reinforce the web by plates or angles.

It should be noted that the combination of a high direct flexural stress and a high direct shearing stress at some point in a beam is apt to occur only under a heavy concentrated load or over a reaction at which the common theory of flexure hardly applies. Moreover, specifications commonly require that the web be reinforced by stiffeners under a heavy concentrated load or over a reaction.

Wooden Beams.—In a wooden beam (if properly sawed) the grain runs parallel to the longitudinal axis of the beam (Fig. 365). Let S_s be the unit shear along the grain at the neutral surface, and S'_s the unit shear along an oblique plane as shown in the figure. In a wooden beam it frequently happens that S'_s is greater than S_s. Now the safe unit shear along the grain is low (say 150 lb./sq. in.) while that across the grain is much higher (say 1000 lb./sq. in.). Hence, since S'_s acts partly across the grain, its safe value is much higher than that for S_s. *That is, in a wooden beam the unit shear along the neutral surface is the significant unit shear.* If a wooden beam fails in shear, it splits along the grain at or near the neutral surface.

On account of the low value of the safe unit shear (along the grain) it is very seldom that the maximum normal stress S' on an oblique plane through a point exceeds the stress in the outer fiber at the section through that point.

* A normal stress on a right section as determined by the flexure formula is called a *direct flexural stress.*

A shearing stress on a right section as determined by the shear formula is called a *direct shearing stress.*

Concrete Beams.—The tensile strength of concrete is very low (relatively). A concrete beam (if not reinforced) will fail therefore in tension. Consider the beam as uniformly loaded (Fig. 366). Let A be a point in the middle right section. There being no vertical shear at the middle section, the maximum

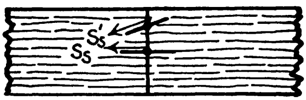


FIG. 365.

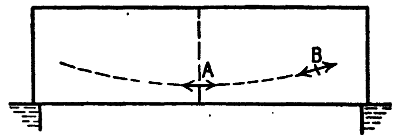


FIG. 366.

tensile stress at A acts normally to the right section as shown in the figure.

Nearer a support (at B, say), there is a vertical shear (and a bending moment also). Hence at B the maximum tensile stress acts obliquely to the right section as shown. Reinforcing bars (steel bars, Fig. 367) are embedded, therefore, in the concrete

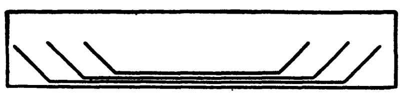


FIG. 367.

with the ends bent upward so that toward the ends of the beam the steel bars tend to follow the lines of maximum tensile stress. In designing concrete beams it is assumed that the steel bars carry the entire tension on the tensile side of the beam.

222. Graphical Solution.—The problem of the previous article may be solved by the following graphical construction. In Fig. 368 lay off OK equal to S (to the right for tension). Lay off KL perpendicular to OK and equal to S_s. Consider the direction of S_s on face bc of Fig. 360 as positive. Bisect OK. With M as a center and ML as radius draw the circle. OP is the maximum normal stress. ON is the minimum normal stress. ML is the maximum shearing stress. The maximum normal stress occurs on a plane parallel to PL and passing through the element. The minimum normal stress occurs on a plane parallel to NL, which is perpendicular to LP. The maximum shearing stress occurs on the perpendicular planes LT and LR.

223. Combined Flexure and Torsion.—A shaft transmitting power is usually subjected to bending and twisting, *i.e.*, to flexure and torsion. Consider the crankshaft shown in Fig. 369. Let P

be the vertical pressure against the crank pin when the arm is in a horizontal position.

If two equal and opposite forces P' and P'', each equal and parallel to P, are introduced at O, the force P acting at B may be replaced by a force P'' (equal to P) acting at O, and a couple PP'

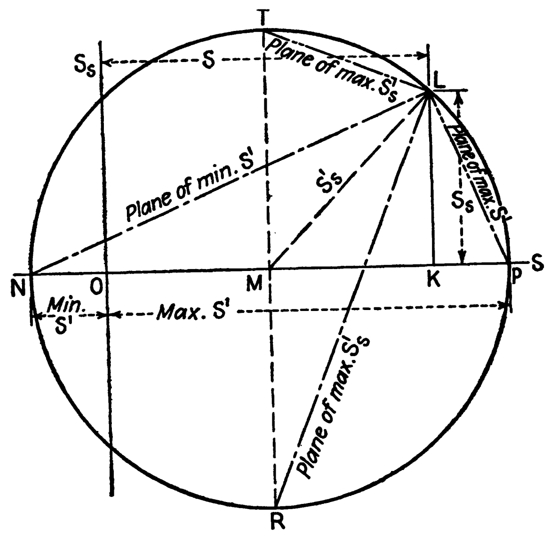


FIG. 368.

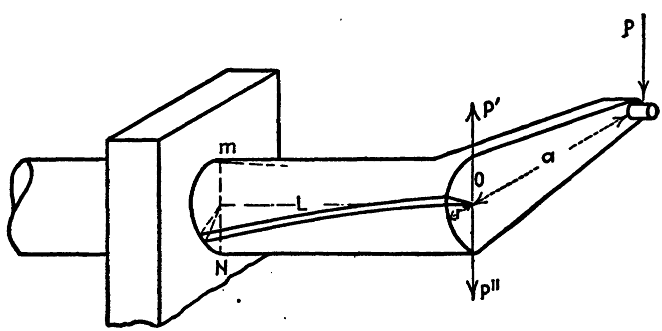


FIG. 369.

whose torque is $T = Pa$ (Art. 99). The force P'' produces a bending moment in the shaft. This moment is a maximum at N and has a value $M_{\text{max.}} = PL$.

The torque T produces shear on a right section. The maximum intensity of this shear occurs on an outer element (say at m) and may be found by means of the torsion formula $T = S_sJ/r$.

That is, since $J/r = \pi r^3/2$.

$$S_s = \frac{Tr}{J} = \frac{2T}{\pi r^3} = \frac{2Pa}{\pi r^3}. \qquad (a)$$

The bending moment M produces a normal stress on the section at N. The maximum intensity of this stress occurs in the outer fiber (at m) and may be found by means of the flexure formula $M = SI/c$. That is, since $I/c = \pi r^3/4$,

$$S = \frac{Mc}{I} = \frac{4M}{\pi r^3} = \frac{4PL}{\pi r^3}. \qquad (b)$$

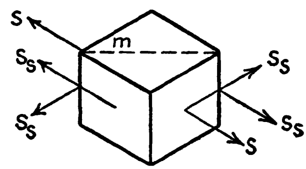


Fig. 370.

The *unit* stresses acting on the outer element m are represented in Fig. 370. The maximum stress acting on an oblique plane (diagonal plane) may be found, therefore, by means of the formulas

$$\left.\begin{aligned} \text{Max. } S_s' &= \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}. \\ \text{Max. } S' &= \frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2} \\ &= \frac{S}{2} + \text{max. } S_s'. \end{aligned}\right\} \qquad (c)$$

Note.—The force P'' acting at O (Fig. 369) produces a vertical shear. The intensity of this shear is a maximum at the neutral axis (on the central horizontal line) and is zero at m (Art. 164). As a rule, the vertical shear *in a shaft* need not be considered. Even at the neutral axis the intensity of the vertical shear is relatively small.

Example I.—In Fig. 369, let $P = 4000$ lb., $a = 6$ in., and $L = 8$ in. Required the radius of a steel shaft if the maximum tensile (or compressive) stress is not to exceed 16,000 lb./sq. in. and the maximum shearing stress is not to exceed 10,000 lb./sq. in.

From Eqs. (a) and (b), respectively,

$$S_s = \frac{2 \times 4000 \times 6}{\pi r^3} = \frac{15{,}270}{r^3}$$

and

$$S = \frac{4 \times 4000 \times 8}{\pi r^3} = \frac{40{,}750}{r^3}.$$

Therefore [Eq. (c)],

$$\text{Max. } S'_s = \sqrt{\left(\frac{20{,}375}{r^3}\right)^2 + \left(\frac{15{,}270}{r^3}\right)^2} = \frac{1}{r^3}\sqrt{20{,}375^2 + 15{,}270^2}$$
$$= \frac{25{,}450}{r^3}.$$

Or, since maximum S'_s is given as 10,000 lb./sq. in.,

$$10{,}000 = \frac{25{,}450}{r^3}.$$

Therefore

$$r = 1.365 \text{ in.}$$

Also, [Eq. (c)],

$$\text{Max. } S' = \frac{1}{r^3}(20{,}375 + 25{,}450).$$

Or, with max. $S' = 16{,}000$,

$$16{,}000 = \frac{1}{r^3}(20{,}375 + 25{,}450).$$

Therefore

$$r = 1.42 \text{ in.}$$

That is, r must be 1.365 in. if the shearing stress is not to exceed 10,000 lb./sq. in., and r must be 1.42 in. if the tensile stress is not to exceed 16,000 lb./sq. in. Hence make the diameter of the shaft the nearest commercial size above 2.84 in. (say $2\frac{7}{8}$ in.).

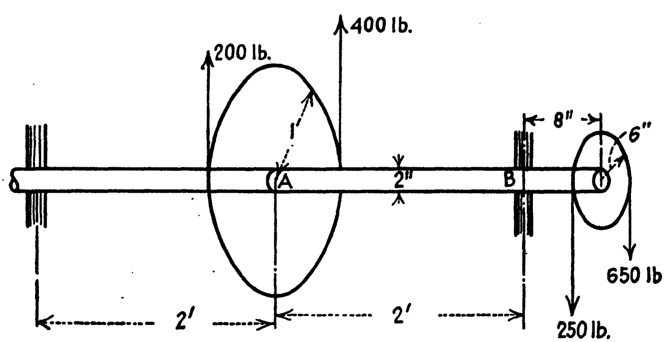


Fig. 371.

Example II.—Two pulleys are mounted as shown in Fig. 371. The diameter of the shaft is 2 in. Required to find the maximum unit tensile and shearing stresses in the shaft.

The maximum bending moment occurs either at A or at B. To find maximum M, deal with the shaft as a beam loaded as

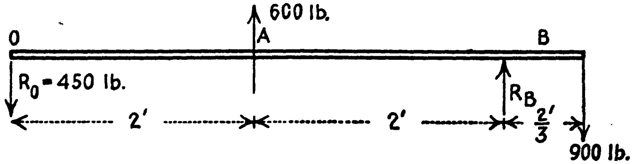


FIG. 372.

shown in Fig. 372. Putting Σ moments about $O = 0$,

$$R_B \times 4 + 600 \times 2 - 900 \times 4\tfrac{2}{3} = 0;$$
$$R_B = 750 \text{ lb.}$$

Putting $\Sigma M_B = 0$,

$$4R_O = 600(2) + 900(\tfrac{2}{3})$$
$$R_O = 450 \text{ lb.}$$
$$M_A = -450 \times 2 = -900 \text{ ft.-lb.}$$
$$M_B = -900 \times \tfrac{2}{3} = -600 \text{ ft.-lb.}$$

Therefore numerically

$$\text{Max. } M = 900 \text{ ft. lb.} = 10{,}800 \text{ in.-lb.}$$

Hence [Eq. (b)]

$$S = \frac{4M}{\pi r^3} = 4 \times \frac{10{,}800}{\pi \times 1^3} = 13{,}750 \text{ lb./sq. in.}$$

The torque is $T = (650 - 250) \times 6 = 2400$ in.-lb.* Hence [Eq. (a)]

$$S_s = \frac{2T}{\pi r^3} = \frac{2 \times 2400}{\pi 1} = 1528 \text{ lb./sq. in.}$$

Therefore

$$\text{Max. } S_s' = \sqrt{6875^2 + 1528^2} = 7040 \text{ lb./ q. in.}$$

and

$$\text{Max. } S' = 6875 + 7040 = 13{,}920 \text{ lb./sq. in.}$$

224. Torsional, Flexural, and Axial Stresses Combined. Fig. 373.—The figure represents a cylindrical prism subjected to a flexural and an axial stress as shown. Let S_1 equal the unit

* Note that (650 − 250) × 6 = (400 − 200) × 12. That is, the shaft is in equilibrium. Equilibrium, however, does not mean "standing still." A body is in equilibrium when it is not accelerating.

stress in the outer fiber due to the bending moment PL, and let S_2 equal the axial stress due to Q. At A the two stresses combine. That is, at A, the stress in the outer fiber is

$$S = S_1 + S_2, \text{ (compression).} \tag{a}$$

Assume now that the prism is subjected also to a torque. Let S_s equal the unit shearing stress on an outer element of the right section AB. Hence at A

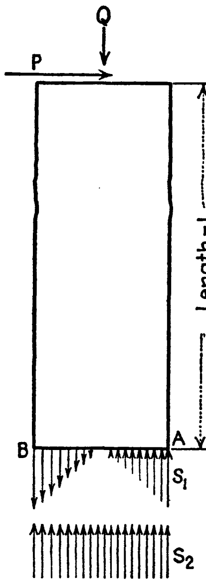


FIG. 373.

$$\text{Max. } S_s' = \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}$$

and

$$\text{Max. } S' = \frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}$$

where $S = S_1 + S_2$.

Note.—Sometimes a shaft is subjected to an axial thrust. For instance, owing to the thrust of the water against a propeller, the propeller shaft is subjected to a thrust and may be subjected also to a bending moment in rough water. Turbines are sometimes fastened to vertical shafts, and the weight of turbine, pulleys, etc., attached to the shaft may produce a thrust (or a tension) that must be considered.

Example.—A shaft 4 in. in diameter (d = 4 in.) is subjected to a bending moment M = 2000 ft.-lb., and to an axial thrust of Q = 24,000 lb. What torque may be applied if max. S_s' = 9000 lb./sq. in. and max. S' = 15,000 lb./sq. in.?

$$S_1 = \frac{4M}{\pi r^3} = \frac{4 \times 2000 \times 12}{\pi 2^3} = 3820 \text{ lb./sq. in.}$$

$$S_2 = \frac{Q}{\pi r^2} = \frac{24{,}000}{\pi 2^2} = 1910 \text{ lb./sq. in.}$$

Therefore

$$S = S_1 + S_2 = 5730 \text{ lb./sq. in.}$$

Consider now the torque [Eq. (a), Art. 223],

$$S_s = \frac{2T}{\pi r^3} = \frac{2T}{\pi 2^3} = 0.0796T.$$

Hence

$$S'_s = 9000 = \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2} = \sqrt{2865^2 + (0.0796T)^2}.$$

Solving for T,

$$T = 107{,}000 \text{ in.-lb.} = 8920 \text{ ft.-lb.}$$

$$S' = 15{,}000 = \frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + S_s^2}$$

$$= 2865 + \sqrt{2865^2 + (0.0796T)^2}.$$

Again solving for T,

$$T = 148{,}100 \text{ in.-lb.} = 12{,}340 \text{ ft.-lb.}$$

Shear governs:

Max. T = 8920 ft.-lb. as the largest permissible value.

PROBLEMS

208. A shaft, 2 in. in diameter, is subjected to torsion and bending. The torque is T = 1000 ft.-lb. and the bending moment is M = 800 ft.-lb. Find the maximum unit stresses (S'_t, S'_c, S'_s).

Ans. S = 12,230; S_s = 7643; max. S'_t = 15,900; max. S'_c = 15,900; max. S'_s = 9790 lb./sq. in.

209. Solve Problem 208 with the following change: in addition to the torque and bending moment the shaft is subjected to an axial pull of 2000 lb.

Ans. S = 12,860; S_s = 7640; max. S'_t = 16,400; max. S'_c = 15,780; max. S'_s = 9990 lb./sq. in.

210. A solid steel shaft is to transmit 240 hp. at 120 r.p.m. The shaft must also sustain a bending moment of 4000 ft.-lb. Given that max. S'_s = 8000 lb./sq. in. and max. S' = 12,000 lb./sq. in., find the radius of the shaft.

Ans. r = 2.21 in.

211. Solve Problem 210 if the shaft is to be a hollow steel shaft with an inside diameter of 4 in. *Ans.* Outside diameter = 5.14 in.

212. Compare the hollow shaft of Problem 211 with the shaft of Problem 210 with respect to the amount of metal they contain.

Ans. A_4/A_3 = 8.19/15.27.

225. Relation between Planes of Maximum Unit Shear and Maximum Unit Normal Stress.—Figure 374 is a repetition of Fig. 360*b* except that the stresses on the faces are represented by their intensities. Let θ'_s designate a value of θ that will make S'_s a maximum (numerically). That is [Eq. (7), Art. 219],

$$\tan 2\theta'_s = \frac{\frac{S}{2}}{S_s}. \qquad (a)$$

In like manner, let θ' designate a value of θ that will make S' a maximum (or a minimum). That is [Eq. (9), Art. 220],

$$\tan 2\theta' = -\frac{S_s}{\frac{S}{2}}. \tag{b}$$

Comparing Eqs. (b) and (a),

$$\tan 2\theta' = -\frac{1}{\tan 2\theta'_s} = -\cot 2\theta'_s \tag{c}$$

$$= \tan (2\theta'_s + 90°), \text{ by trigonometry.}$$

Therefore

$$2\theta' = 2\theta'_s + 90°$$

or

$$\theta' = \theta'_s + 45°.$$

Hence (Fig. 375) *a plane of maximum (or minimum) unit normal stress makes an angle of* 45° *with a plane of maximum unit shear.*

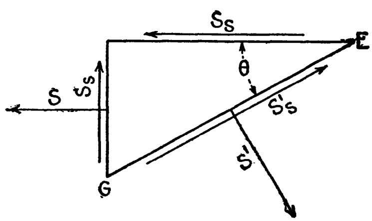


Fig. 374.

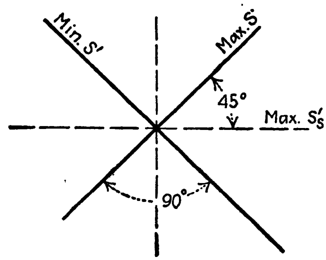


Fig. 375.

226. Principal Planes. Principal Stresses.—In Art. 217, it was shown that if the stresses at a point on two mutually perpendicular planes are *simple* normal stresses (no shear), then one stress is the normal stress of maximum intensity and the other is the normal stress of minimum intensity at that point. It will now be shown that *the unit shear at a point always is zero on a plane of maximum or minimum unit normal stress.*

With reference to Fig. 374, the unit shear on the diagonal plane *EG* is [Art. 218, Eq. (5)]

$$S'_s = \frac{S}{2}\sin 2\theta + S_s \cos 2\theta.$$

Putting $S'_s = 0$,

$$\tan 2\theta = -\frac{S_s}{\left(\frac{S}{2}\right)}. \tag{a}$$

Equation (*a*) is satisfied by two values of θ that are 90° apart. Hence, at a point in a body, the unit shear is zero on each of two mutually perpendicular planes. Now Eq. (*a*) also gives the planes of maximum and minimum unit normal stress [Eq. (9), Art. 220]. That is, a plane of zero shear is a plane of maximum or minimum unit normal stress.

A similar investigation of an elementary prism on whose *six* faces act stresses (normal, shearing, or both) leads to the conclusion that at a point in a body *three* planes at right angles to each other can always be found on which no shearing stresses act. The three planes of no shear are called the *principal planes* at that point, and the simple normal stresses acting on these planes are called the *principal stresses* at that point. Moreover, *a normal stress of maximum or minimum intensity always is a principal stress.*

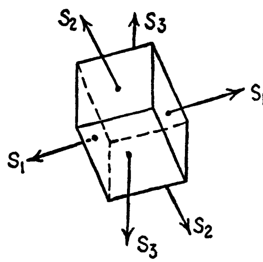


FIG. 376.

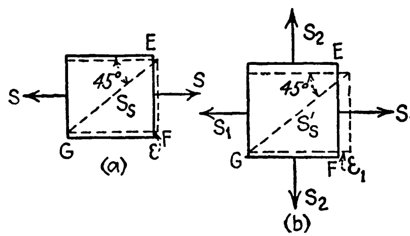


FIG. 377.

The results of this article may be stated in another way. At a point in a body under stress there is always an elementary prism on whose faces *only* normal stresses act (Fig. 376). The normal stresses acting on the faces of this prism are the principal stresses at that point. One stress is the normal stress of maximum intensity, and one is the normal stress of minimum intensity at that point. A principal stress may be tensile (+) or compressive (−).

In the engineering problems that commonly occur, at least one of the principal stresses is zero. The assumption of such zero stress will be made in the remaining articles of this chapter.

227. Theory of Failure.—Consider two unit cubes of a material (cubes of unit dimensions), and assume that the two cubes have the same physical properties. Since the cubes are unit cubes, the stresses on their faces are unit stresses (Fig. 377). For the first cube (Fig. 377*a*), S is the unit axial stress, S_s is the

unit shearing stress along the diagonal plane EG, and ϵ is the unit longitudinal strain. For the second cube (Fig. 377*b*), S_1 and S_2 are unit biaxial stresses, S'_s is the unit shearing stress along the diagonal plane, and ϵ_1 is the unit longitudinal strain in the direction of the maximum normal stress (assuming $S_1 > S_2$).

Note that S (Fig. 377*a*) and S_1 and S_2 (Fig. 377*b*) are principal stresses (Art 226). Note also that S_s or S'_s is numerically greater than the unit shearing stress on any other plane through E. (A plane of maximum unit shear makes an angle of 45° with the plane of the maximum or minimum unit normal stress, Art. 225.) Moreover [Art. 217, Eq. (4)],

$$\text{Max. } S_s = \frac{S}{2}; \qquad \text{max. } S'_s = \frac{S_1 - S_2}{2}. \qquad (a)$$

Assume that the elastic limit is reached in both prisms. A very important question now arises. At the elastic limit, is $S_1 = S$ or $\epsilon_1 = \epsilon$ or $S'_s = S_s$? That is, does the maximum unit normal stress or the maximum unit longitudinal strain or the maximum unit shearing stress determine the safety of the material? This question has led to three theories.

1. *Maximum Normal Stress Theory.*—According to this theory, the elastic limit is reached when the maximum unit normal stress reaches a certain value, irrespective of the other stresses that may act. For instance, if in Fig. 377*a* the elastic limit is reached when $S = 35{,}000$ lb./sq. in., then the elastic limit in Fig. 377*b* will be reached when $S_1 = 35{,}000$ lb./sq. in. irrespective of the value of S_2 (S_1 being greater than S_2).

2. *Maximum Strain Theory.*—According to this theory, the elastic limit will be reached when the unit strain in the direction of the maximum unit normal stress reaches a certain value, irrespective of how this strain is produced. For instance, if in Fig. 377*a* the elastic limit is reached when $\epsilon = 0.002$, then the elastic limit will be reached in Fig. 377*b* when $\epsilon_1 = 0.002$ irrespective of how this strain is produced.

3. *Maximum Shear Theory.*—According to this theory, the elastic limit will be reached when the maximum unit shear reaches a certain value, *i.e.*, when $S'_s = (S_1 - S_2)/2$ reaches a certain value. For instance, if in Fig. 377*a* the elastic limit is reached when $S_s = 20{,}000$ lb./sq. in., then the elastic limit in Fig. 377*b* will be reached when $S'_s = 20{,}000$ lb./sq. in. irrespective of how this stress is produced.

It is a very common occurrence for a material to fail in shear. Figure 378 illustrates how a material subjected to simple tension or compression is apt to fail. In some cases, however, the maximum shear theory does not seem to hold when a material is subjected to a simple normal stress. For instance, if a bar of cast iron (Fig. 378*d*) is subjected to simple tension, the plane of the fracture is at right angles to the axis of the bar. This seems to indicate that cast iron *under simple tension* will fail according to the maximum normal stress theory or according to the maximum strain theory.

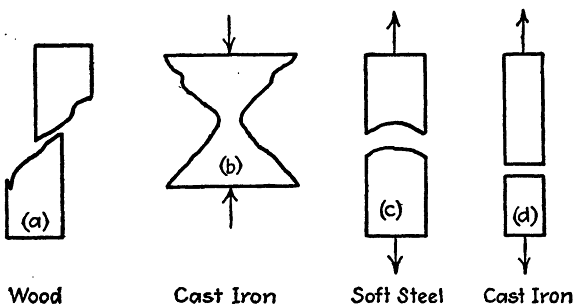


FIG. 378.

Note.—For a fourth theory based on maximum strain energy, see texts on advanced Mechanics of Materials.

Experiments to determine the true cause of failure are difficult to make, and such experiments as have been made are not consistent. The more recent experiments, however, seem to indicate that *a material will fail either in shear* according to the maximum shear theory *or in strain*, according to the maximum strain theory, *depending upon which of the two, shear or strain, reaches its limiting value first.** If a plate of a material is subjected to increasing biaxial stresses (Fig. 377*b*), the shear along the diagonal plane is

$$S'_s = \frac{S_1 - S_2}{2}.$$

If S_2 is nearly equal to S_1 (and of the same sign), *i.e.*, if S_1/S_2 is nearly equal to unity, S'_s is small and the strain ϵ_1 will reach its limiting value first. Failure then will occur according to the maximum strain theory. If the ratio S_1/S_2 is increased, *i.e.*, if S_1 is large in relation to S_2, S'_s is increased. If S'_s reaches

* BECKER, A. J., Univ. Illinois Expt. Sta., *Bull.* 85.

its limiting value before ϵ_1 reaches its limiting value, failure will occur in accordance with the maximum shear theory. It should be remembered, however, that these experiments were limited in their scope and that, therefore, it is not safe to conclude that the maximum shear or maximum strain theory is applicable to all materials or under all conditions of loading. The only conclusions that can safely be reached at present are

1. In many cases, the maximum shear theory can hardly be questioned.

2. In some cases, the maximum strain theory seems the most probable. In the design of heavy guns and in similar problems, the maximum strain theory is extensively used.

228. Applications of Maximum Strain Theory.—Referring to Fig. 377*a*,

$$E = \frac{S}{\epsilon} \text{ (Art. 26).}$$

Therefore

$$\epsilon = \frac{S}{E}. \tag{a}$$

In like manner (Fig. 377*b*), if S_1 acts alone,

$$\epsilon_1' = \frac{S_1}{E}.$$

If m = Poisson's ratio (Art. 22), the lateral contraction due to S_2 (acting alone) is

$$\epsilon_1'' = \frac{mS_2}{E}.$$

Hence, putting $\epsilon = \epsilon_1 = \epsilon_1' - \epsilon_1''$,

$$\frac{S}{E} = \frac{S_1}{E} - \frac{mS_2}{E}.$$

Or

$$S = S_1 - mS_2. \tag{b}$$

That is, the simple axial stress S will, if acting alone, produce the same unit longitudinal strain as is produced by the biaxial stresses S_1 and S_2 acting simultaneously. The stress S is called the *equivalent simple unit stress.* Accordingly, the maximum strain theory may be interpreted as meaning that S, the *equivalent simple stress, may be taken as the criterion of safety.* That is, the prism is safe against injury if the equivalent simple stress S

does not exceed a certain value, this value of S being determined experimentally by subjecting a prism of the material to a simple axial stress (Fig. 377*a*).

Example I.—Let $S_1 = 2S_2$ (stresses in a boiler shell, say). If the allowable stress in simple tension is $S = 16{,}000$ lb./sq. in., and the allowable unit shear is $S_s = 10{,}000$ lb./sq. in., find the maximum safe value of S_1 if Poisson's ratio is $m = 0.30$.

On the basis of strain [Eq. (*b*)],

$$16{,}000 = S_1(1 - 0.30 \times \tfrac{1}{2}) = 0.85S_1.$$

Therefore

$$S_1 = 18{,}800 \text{ lb./sq. in.}$$

On the basis of shear,

$$10{,}000 = \frac{S_1 - \dfrac{S_1}{2}}{2} = \frac{1}{4}S_1.$$

Therefore

$$S_1 = 40{,}000 \text{ lb./sq. in.}$$

Hence strain governs, and $S_1 = 18{,}800$ lb./sq. in.

Example II.—Take the same data as in Example I except that S_2 is a compressive stress and S_1 is a tensile stress.

$$16{,}000 = S_1(1 + 0.30 \times 0.50) = 1.15S_1.$$

Therefore

$$S_1 = 13{,}900 \text{ lb./sq. in.}$$

Also

$$10{,}000 = \frac{\left(S_1 + \dfrac{S_2}{2}\right)}{2} = 0.75S_1.$$

Therefore

$$S_1 = 13{,}330 \text{ lb./sq. in.}$$

In this case shear governs and $S_1 = 13{,}330$ lb./sq. in. Note that S_1 is less than 16,000.

229. The Ordinary Maximum Stress Theory.—When a material is under a simple axial stress, the theory of failure is of little practical importance. For instance, in a soft steel rod under simple tension, incipient failure seems to be due to shear. Theoretically, since $S_2 = 0$ [Eq. (*a*), Art. 227], the maximum unit shear is

$$S_s = \frac{S}{2}$$

and the longitudinal strain at the elastic limit is [Eq. (a), Art. 228]

$$\epsilon = \frac{S}{E}.$$

It is immaterial, therefore, which of the three, $S(= 2S_s)$, $S_s(= S/2)$, or $\epsilon(= S/E)$, is taken as the criterion for the elastic limit of the material. When a material is subjected to biaxial (or to triaxial) stresses, the situation is very different, since then the unit shear S'_s and the unit strain ϵ_1 depend largely upon the ratio S_1/S_2.

In practice, it is assumed in all ordinary cases that failure occurs when some unit stress, shearing, tensile, or compressive, exceeds its ultimate value. That is, in practice, it is usually assumed that failure is due to the maximum normal stress or to the maximum shearing stress, depending on which of the two, normal or shearing, reaches its ultimate value first. So long as the true cause of failure is not satisfactorily explained, it seems wise to continue this procedure. It may be added that, except in unusual cases, the ordinary maximum stress theory gives results fairly consistent with the results of experiments.

PROBLEMS

213. A steel shaft 2 in. in diameter is subjected to a bending moment of 600 ft.-lb. and a torque of 300 ft.-lb. Compute the resulting maximum shearing and tensile stresses.

Ans. $S'_s = 5130$ lb./sq. in.; $S' = 9720$ lb./sq. in.

214. In Problem 213, determine the directions of the planes of maximum shear and of maximum normal stress. For S'_s, 31°44′ and 121°44′; for S', 76°44′ and 166°44′.

215. The shaft of Problem 213 is to be subjected also to a total tension P in addition to the other actions mentioned. Compute P if the allowable resulting shearing stress $S'_s = 10{,}000$ lb./sq. in. *Ans.* $P = 32{,}300$.

216. Find the induced maximum normal stress in Problem 215:

Ans. $S' = 19{,}730$ lb./sq. in.

217. A shaft 3 in. in diameter is subjected to a torque of 2000 ft.-lb. What bending moment can the shaft sustain in addition to the torque if $S'_s = 10{,}000$ lb./sq. in.? *Ans.* 47,300 in.-lb.

218. In Problem 217, what normal stress is induced by the combination of the torque and the bending moment? *Ans.* 18,900 lb./sq. in.

219. A shaft is subjected to a torque T and a bending moment M. Prove that the maximum shearing stress S'_s and the maximum normal stress S' may be written, respectively, as

$$S'_s = \frac{r}{J}\sqrt{M^2 + T^2} = \frac{T'r}{J}.$$

where T' (equaling $\sqrt{M^2 + T^2}$) is called the equivalent torque; and

$$S' = \frac{c}{I}\,\frac{M + \sqrt{M^2 + T^2}}{2} = \frac{M'c}{I}$$

where M' [replacing $(M + \sqrt{M^2 + T^2})/2$] is called the equivalent bending moment.

Suggestion.—For torsion $S_s = Tr/J = Tr/2I$; for bending $S = Mc/I = Mr/I$.

220. Design a solid cylindrical steel shaft to transmit 200 hp. at 100 r.p.m. The shaft is subjected to a bending moment of 4000 ft.-lb. Take $S' = 16{,}000$ lb./sq. in. and $S'_s = 10{,}000$ lb./sq. in. Use the equations of Problem 219. *Ans.* d required by shear = 4.09 in.

221. A hollow shaft whose inside diameter is 3 in. and whose outside diameter is 6 in. must carry a bending moment of 20,000 ft.-lb. What horsepower can this shaft transmit at 90 r.p.m. if S'_s is 9000 lb./sq. in. and S' is 15,000 lb./sq. in.? *Ans.* 376 hp.

222. On the basis of the difference in shearing resistance across the grain and parallel to the grain in wood, explain why twisting a cylindrical piece of wood with the grain parallel to the axis of the piece will cause it to split lengthwise.

223. A cylindrical piece of wood with the grain parallel to the axis of the cylinder had a diameter of $\frac{1}{2}$ in. In a testing machine it failed under a torque of 9 in.-lb. Compute the ultimate shearing stress parallel to the grain. *Ans.* 367 lb./sq. in.

224. At a point in a member, the horizontal tensile stress is 16,000 lb./sq. in. and the vertical shearing stress is 3430 lb./sq. in. Compute the angles locating the principal planes. *Ans.* 78.4° and 168.4°.

CHAPTER XI

NONPRISMATIC AND SPECIAL BEAMS

230. Preliminary Remarks.—In addition to the ordinary cases of prismatic beams, and yet subject to the same general methods of analysis used with such beams, there are several other types of beams of great practical importance. Nonprismatic or tapering beams, reinforced beams, curved beams, and flat plates are examples of these types.

As in the case of prismatic beams, the determination of their strength is of prime importance. This requires the finding of the maximum bending stress. This stress is dependent upon the span, kind of loading, the material in the beam, and particularly the dimensions of the cross-sections.

Aside from strength, there is also the usual consideration of deflection as in the case of prismatic beams; also the bearing area, internal shear, etc., may need examination.

The purpose of this chapter is to take up a few cases of several types of beams other than prismatic beams; to show how they are related in their analysis to simpler beams; and to give certain useful methods and results.

231. Dangerous Section in a Nonprismatic Beam.—For a prismatic beam, the moment diagram is also a diagram of fiber stresses (Art. 144) and its importance is due to the fact that it shows the variation of bending stress as well as the variation of moment at all the sections of the beam. The maximum moment occurs at the section of greatest bending stress and this section is consequently the dangerous section.

In the case of a prismatic beam, under a fixed loading, the maximum fiber stress is directly proportional to the maximum moment in the beam. That is, with $S = Mc/I$, since c/I is the same for all sections, S is directly proportional to M. With sectional dimensions varying, it is clear that the moment of inertia of the section, *viz.*, I, as well as the distance to the outer fiber c must vary. The fiber stress in such beams is not proportional to the moment alone but must be found from the equa-

tion $S = Mc/I$, in which all of the quantities on the right of the sign of equality may have different values for different sections of the beam. *The dangerous section obviously occurs where the unit fiber stress is a maximum.*

The moment diagram, while still of some use, is not in such a case a graph of the stress due to bending. A separate stress diagram should be constructed if it is needed. Usually, however, such diagrams are not constructed, since it is not difficult to perceive the dangerous section after a few typical cases have been studied and the methods of analysis are understood.

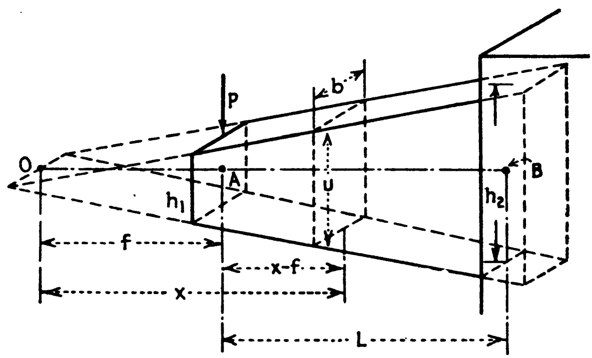


FIG. 379.

If it is assumed that local concentrations of stress are neglected, a beam which is apparently prismatic, if holes or notches are made in it, might become a case in which the dangerous section is not the section at which the maximum moment is found.

232. Dangerous Section in a Truncated Wedge. Cantilever Beam with End Load P. Uniform Thickness b.—Neglect the weight of the beam. Continue the sloping surfaces until they meet in a line whose mid-point is O. Let f be the distance from the free (loaded) end of the cantilever to the point O. Let the variable height of the beam at any intermediate section along the beam be u, the width being of constant value b.

From similar triangles (Fig. 379),

$$\frac{u}{h_1} = \frac{x}{f}.$$

Consequently

$$\frac{I}{c} = \frac{bu^2}{6} = \frac{bh_1^2x^2}{6f^2}.$$

The moment at the section is $P(x - f) = Px - Pf$. Putting

$$Px - Pf = \frac{SI}{c} = \frac{Sbh_1^2x^2}{6f^2},$$

and solving for S,

$$S = \frac{6Pf^2x}{bh_1^2x^2} - \frac{6Pf^3}{bh_1^2x^2}$$
$$= \frac{6Pf^2}{bh_1^2}\left(\frac{1}{x} - \frac{f}{x^2}\right).$$

The value of S will be a maximum for that value of x which will make dS/dx equal to zero.

$$\frac{dS}{dx} = \frac{6Pf^2}{bh_1^2}\left(-\frac{1}{x^2} + \frac{2f}{x^3}\right) = 0.$$

Therefore

$$x = 2f.$$

Stated in words, this result is as follows: If a truncated wedge of constant width is used as a cantilever with a concentrated load at the end, the dangerous section will be as far from the free end A (Fig. 379) as the point O of the completed wedge is on the other side. The moment at the dangerous section will be $M = Pf$ and the depth of the beam will be $2h_1$. The section modulus at this section is

$$\frac{I}{c} = \frac{bu^2}{6} = \frac{b(2h_1)^2}{6} = \frac{4bh_1^2}{6}.$$

Therefore

$$S = \frac{Mc}{I} = \frac{3}{2}\frac{Pf}{bh_1^2}.$$

If f should exceed L in length, that fact would imply the existence of a dangerous section beyond the face B of the support. Since a dangerous section cannot occur within the wall in any of the well-protected sections of the beam, the maximum stress is found in the section at the face B of the wall into which the beam is built. The bending moment at section B is then PL. The stress computed as in the case above is

$$S_B = \frac{Mc}{I} = \frac{6PL}{bh_2^2}.$$

The results of the foregoing analysis may readily be adapted to the case of a double truncated wedge-shaped beam on end supports, bearing a concentrated load at the section which forms the common base of the two wedges (Fig. 380). Each of the wedges increases in size from the support toward the load point. The two portions of the beam are not necessarily alike geometrically. Each portion may be viewed as a cantilever beam under the bending action of the reaction of the support as the end load. Then, from the preceding case of a cantilever, the dangerous section may be located on each portion as far from its support as the distance to the edge of the wedge if the surfaces are extended to an intersection.

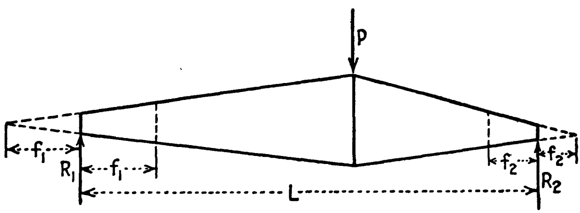


FIG. 380.

Example.—A wooden beam has a constant thickness b equal to 6 in. perpendicular to the side surface shown in Fig. 379. Given $h_1 = 4$ in., $h_2 = 12$ in., $L = 48$ in., $P = 3200$ lb., find the bending stress in the outer fiber at the dangerous section. By similar triangles,

$$\frac{f}{h_1} = \frac{L}{h_2 - h_1}.$$

Using the given numerical values,

$$f = \frac{4 \times 48}{8} = 24 \text{ in.}$$

This is the distance from O to the load P and is also the distance from P to the dangerous section.

At the dangerous section, $u = 8$ in., $b = 6$ in.; then the section modulus

$$\frac{I}{c} = \frac{bu^2}{6} = 64 \text{ in.}^3$$

From the moment equation,

$$Pf = S\frac{I}{c}.$$

Therefore

$$S = \frac{3200 \times 24}{64} = 1200 \text{ lb./sq. in.}$$

For comparison as well as a partial check on the result just obtained, the bending stress at section B adjacent to the wall will be computed. The height of the beam at this section is 12 in. and the computed section modulus is 144 in.^{3}
Then

$$S_B = \frac{3200 \times 48}{144} = 1077 \text{ lb./sq. in.}$$

This value is considerably less than the unit stress computed for the dangerous section.

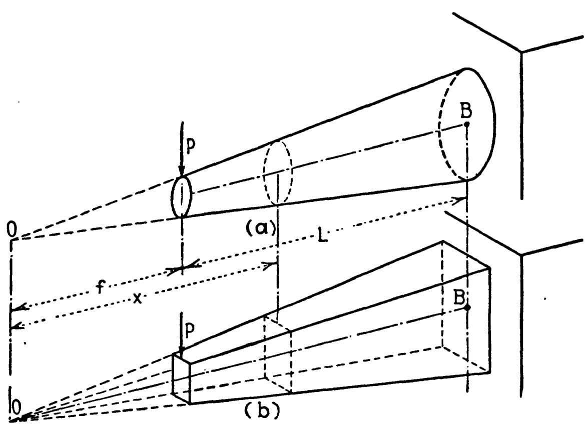


FIG. 381.

233. Truncated Cone or Pyramid as a Cantilever Beam. End Load P. Weight of Beam Neglected.—Figure 381*a* shows a truncated cone used as a cantilever beam, and Fig. 381*b* shows a truncated pyramid. All right sections of either the cone-shaped beam or of the truncated pyramid are geometrically similar.

As in the case of the truncated wedge, extend the edges of the beams to an intersection at the vertex O, which will be taken as the origin of coordinates. Let f be its distance from the loaded free end and let x be the distance from O to any section of the beam between the limiting values f and $f + L$.

The moment of the load about a section at a distance x from O is

$$M = P(x - f) = Px - Pf.$$

For equilibrium, this moment must equal the resisting (internal) moment whose value is

$$M = \frac{SI}{c}.$$

Since the moment of inertia I is expressed in linear units raised to the fourth power and since c is a linear dimension, *the I/c values for two geometrically similar sections are in the ratio of the cubes of similar dimensions.* Whether measured vertically, horizontally, or in any inclined direction, these dimensions, by similar triangles, are in the direct ratio of the distances of the sections from the vertex O.

Let the I/c value at distance x from O be designated Z, and the value at the support B, distant $f + L$ from O, be Z_1. From the geometric similarity,

$$\frac{Z}{Z_1} = \frac{x^3}{(f + L)^3},$$

or

$$Z = \frac{Z_1 x^3}{(f + L)^3}.$$

Accordingly,

$$M = Px - Pf = SZ = \frac{SZ_1 x^3}{(f + L)^3}.$$

Therefore

$$S = \frac{P(f + L)^3(x - f)}{Z_1 x^3},$$

$$= \frac{P(f + L)^3}{Z_1}(x^{-2} - fx^{-3}).$$

To find the maximum value of S, this equation is to be differentiated with respect to x, and the derivative equated to zero. It should be noted that P, Z_1, and $f + L$ are constants and will disappear from the equation when dS/dx equals zero and may accordingly be omitted while differentiating.

Then

$$\frac{dS}{dx} = -2x^{-3} + 3fx^{-4} = 0$$

for a maximum S. Solving,

$$x = \frac{3f}{2},$$

$$= f + \frac{f}{2}.$$

This value of x locates the section of maximum bending stress with respect to the origin O.

Stated in words, the dangerous section of a truncated cone or pyramid (of any shape of cross-section) will be found at a distance from the free (loaded) end equal to one-half of the distance from that end to the vertex, the solid beam being conceived to be extended to the vertex O. This result does not apply to other types of loading.

If $f/2$ is greater than L, it necessarily follows that the maximum bending stress will occur at the section B adjacent to the support.

As in Art. 232, the result obtained may be used in the case of a double truncated cone or pyramid (Fig. 382), loaded at the com-

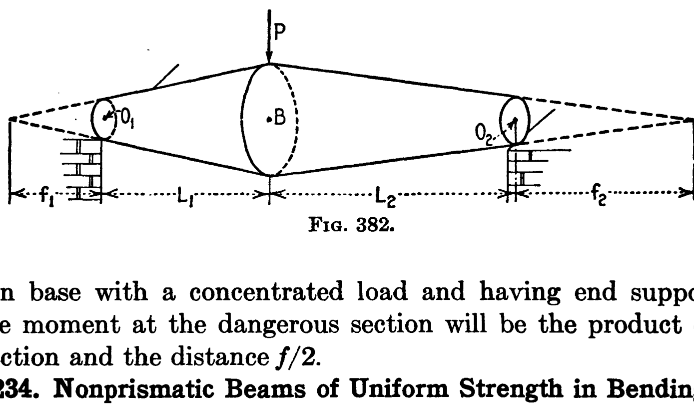


FIG. 382.

mon base with a concentrated load and having end supports. The moment at the dangerous section will be the product of a reaction and the distance $f/2$.

234. Nonprismatic Beams of Uniform Strength in Bending.— The preceding articles suggest that it would be desirable to design beams as to their loading, support, and the varying dimensions of their cross-sections so that the outer fiber stress in bending would have an allowable maximum value at all sections of the beam. This stress would then be uniform in the outer fibers but would, of course, be of lesser value on all interior fibers. A beam designed so that the outer fiber stress is constant (the same for all) is termed a *beam of uniform strength.*

It is not possible at the same time to make the design involve uniform unit internal shear of the allowable value for the material. Therefore, the greatest measure of economy of design cannot be attained. Nevertheless, a considerable saving of material may be made, and the resulting shape of the beam may be more attractive in appearance if beams of uniform strength are employed. Plate girders, reinforced-concrete bridges, levers, and many

parts of small tools illustrate the tendency in industry to adopt designs including beams of uniform strength.

The next articles present methods of solution and illustrative examples of beams of uniform strength in bending.

The weight of the beam itself will be neglected. The method of solution requires the design of a section of the beam at the point of maximum moment. All other sections are then obtained by making the stress in the outer fiber equal to that previously selected or computed at the section of maximum moment.

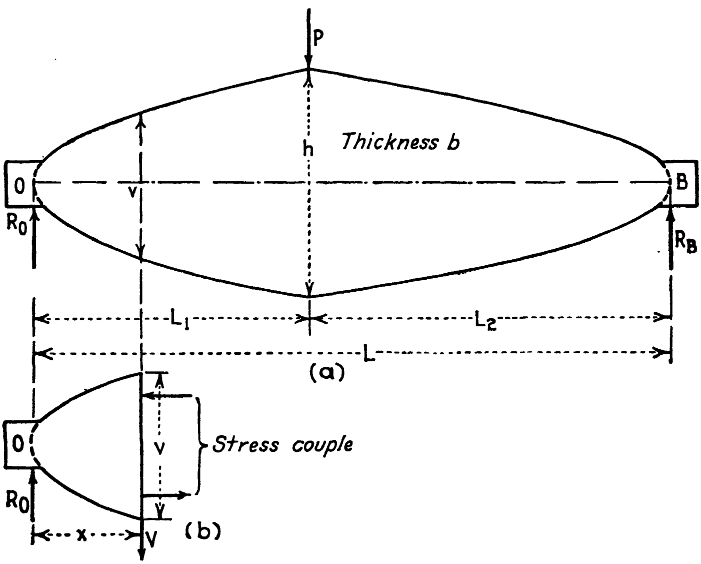


FIG. 383.

Modifications are usually necessary at sections where the internal unit shear governs the dimensions of the beam.

235. Beam of Constant Thickness b. Unsymmetrical Concentrated Load P. Parabolic Form. Fig. 383.—Assume all sections of the beam to be rectangular and of constant width b. The heights of the sections vary. Let the height at any section be v.

The value of R_O is found by taking the sum of the moments about B for the entire beam as a free body. $R_O = PL_2/L$ (Fig. 383a). The value of h for the section under the load P is found by putting

$$\frac{S'I}{c} = R_O L_1 = \frac{S'bh^2}{6}.$$

Therefore

$$S' = \frac{6R_OL_1}{bh^2}. \tag{a}$$

Next, the expression for the outer fiber stress at any other section distant x from O is obtained, involving the variable height v of the section (Fig. 383b). For the free body in this figure which shows the portion of the beam from the origin O to the section at distance x from O, the internal moment

$$\frac{SI}{c} = \frac{Sbv^2}{6} = R_Ox,$$

the last term of the equation being the expression for the moment of the external reaction.

Solving,

$$S = \frac{6R_Ox}{bv^2}. \tag{b}$$

For the beam to have uniform strength, $S = S'$.

Or,

$$\frac{6R_Ox}{bv^2} = \frac{6R_OL_1}{bh^2}.$$

Then

$$\frac{v^2}{h^2} = \frac{x}{L_1},$$

and this may be written

$$\frac{\left(\frac{v}{2}\right)^2}{\left(\frac{h}{2}\right)^2} = \frac{x}{L_1}. \tag{c}$$

Equation (c) shows that the outline of the beam from O to the load is a parabola with the vertex at O.

The parabolic outline for the right-hand portion of the beam is obtained similarly.

236. Shear and Bearing on Beams of Uniform Strength.—To make allowance for shear and bearing, modifications must be made at the ends of the beam as is indicated in Fig. 383. Consider the left end O and assume that the modified end has a rectangular section of width b_O (perpendicular to the paper) and depth h_O.

Shear.—Since the maximum unit shear in a rectangular section of a beam is 1.5 times the average shear [Eq. (12), Art. 164],

the required dimensions of the section at the end may be determined from the equation

$$S_s = \frac{3}{2}\frac{R_O}{b_O h_O}$$

where S_s is the allowable unit shear.

Bearing.—The bearing area needed for the reaction R_O (Fig. 383) is determined from the equation

$$S_c = \frac{R_O}{a_O b_O}$$

where S_c is the allowable unit bearing stress (of the weaker material of beam or support). Hence the modified end must extend to the left of O a distance $a_O/2$ as is indicated in the figure.

In a similar manner, modification of the right end of the beam must also be made.

Problem 225.—Design a cast-steel beam having varying rectangular section. The beam has a thickness of 4 in. and carries a central concentrated load of 36,000 lb. on a span of 8 ft. The unit stress in bending is to be the same for all sections, equal to 18,000 lb./sq. in. Find the height at the section under the load and determine the equation of the contour of the beam with respect to an origin at the end. Find also the necessary depth at the end for shearing using 10,000 lb./sq. in. and taking the maximum unit shear equal to 1.5 times the average shear.

Ans. h = 8.5 in., $2v^2 = 3x$, and, at the ends, h_O = 0.675 in. (minimum).

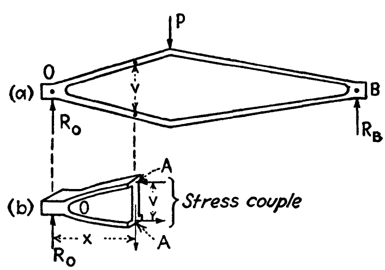


FIG. 384.

237. Cast Beam of I-section. Uniform Strength. Fig. 384.

The beam carries a single concentrated load and has end supports. Assuming that the flange areas are the same at all sections and that they carry all of the bending stress while the web carries all of the shear, the internal or resisting moment is

$$M = SA \times v. \qquad (a)$$

This must equal the bending moment which has a value

$$M = R_O x \text{ (Fig. 384}b\text{)}. \qquad (b)$$

Equating the two values,

$$v = \frac{R_O}{SA}x = \text{constant} \times x. \qquad (c)$$

Equation (c), in the final form, may be written thus since the unit stress is to be the same at all sections and it is evident that R_O and A are also constants. The portion of the beam from P to B may be similarly analyzed.

Equation (c) is the equation of a straight line and therefore the top and bottom surfaces should be plane surfaces.

238. Beam of Uniform Strength on End Supports. Single Eccentric Load. Constant Height h. Fig. 385.—Let the width

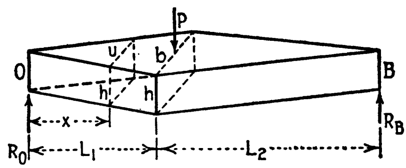


FIG. 385.

at the load section equal b. The stress at this section is found from the equation

$$R_O L_1 = \frac{S'bh^2}{6}, \tag{a}$$

or

$$S' = \frac{6R_O L_1}{bh^2}.$$

The stress at any section, distant x from O, and having a variable width u, is

$$S = \frac{6R_O x}{uh^2}. \tag{b}$$

Equating S' and S,

$$\frac{L_1}{b} = \frac{x}{u}, \qquad \text{or} \qquad u = \frac{b}{L_1}x.$$

This equation shows that u increases directly with x and therefore the beam must be wedge shaped. The ends and the load point would have to be designed for local stresses.

239. Beam of Uniform Strength on End Supports. Similar Rectangular Sections. Single Eccentric Load. Fig. 386.—The load P causes reaction R_O at O and R_C at C, where the end supports are placed. Let O be the origin, x locating any vertical section of varying height $2v$ and width $2u$ in the range O to B. At the load point, the necessary section is a rectangle of height $2h$ and of width $2b$, the coefficient 2 being of convenience in later algebraic work.

The similarity of sections results in the relation

$$\frac{u}{v} = \frac{b}{h}. \tag{a}$$

For the portion OB of the beam,

$$S' = \frac{6R_OL_1}{2b(2h)^2}. \tag{b}$$

For the portion of length x,

$$S = \frac{6R_Ox}{2u(2v)^2}. \tag{c}$$

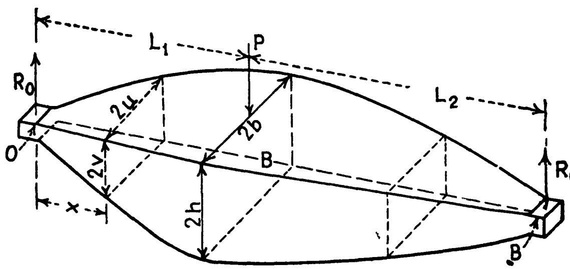


FIG. 386.

Equating S and S' for uniform strength,

$$\frac{x}{uv^2} = \frac{L_1}{bh^2}. \tag{d}$$

This may be written

$$uv^2 = \frac{bh^2}{L_1}x.$$

From Eq. (a), this becomes

$$\left.\begin{aligned} v^3 &= \frac{h^3}{L_1}x. \\ \text{or} \qquad u^3 &= \frac{b^3}{L_1}x. \end{aligned}\right\} \tag{e}$$

The two equations [Eqs. (e)] show that the median curves are cubic parabolas.

Problem 226. Fig. 387.—Required to find how the width should vary in a beam of uniform strength, the height being constant, the sections rectangular, and the load uniformly distributed. The weight of the beam is neglected.

Ans. $u = 4b(Lx - x^2)/L^2$.

240. Reinforcing of Girders for Approximate Uniform Strength. In the effort to economize on material, not only have beams been designed with special outlines and cross-sections, but beams of prismatic section have been reinforced by the addition of plates, angles, or other shaped pieces so that their resistance in bending would conform more nearly to the change in the bending moment of the external loads. If bridge trusses, viewed in a broad way, are considered as skeletonized beams, it will be observed that the most economical modern types have material disposed in the various sections in close agreement with the arrangement most favorable to uniform strength. Modern repair work on steel structures by means of welding is quite

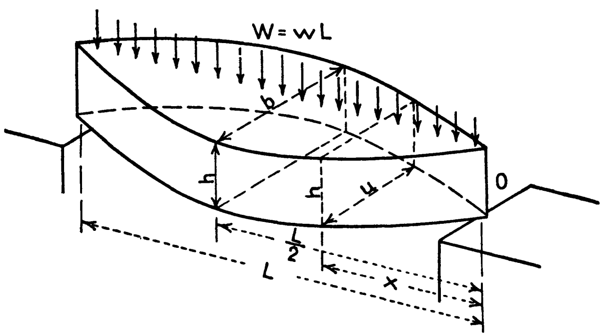


FIG. 387.

commonly in accord with a revised design aiming at more uniform stress distribution.

In the design of plate girders, the most obvious plan to follow in increasing the section modulus of the beam at the necessary points is to rivet cover plates at the top and at the bottom flanges. A similar but simpler problem arises when a beam of prismatic section is to be reinforced with added plates, angles, or bars with the idea of modifying the beam to make it more nearly a beam of uniform strength. Reinforcing of a heavy I-beam with flange plates is the topic of the following article.

241. Design of Reinforcing Plates for an I-beam.—A WF steel I-beam (Fig. 388) has a nominal depth of 36 in. and weighs 300 lb./ft. The moment of inertia about the centroidal axis perpendicular to the web is 20,290 in.^{4} and the section modulus is 1105.1 in.^{3} The actual depth of the beam is 36.72 in., the area of the section is 88.17 sq. in., and the width of the flanges is 16.65 in.

Assume that the beam has end piers. The applied load, exclusive of the weight of the beam, is 180 tons. Let the allowable unit stress in bending be 18,000 lb./sq. in. The span is 60 ft. In addition to the external load of 6000 lb./ft., it is necessary to consider the weight of the beam itself, making the total load per foot 6300 lb. The reinforcing plates, when added, will increase the load, but their effect in bending will not be included in the following calculations.

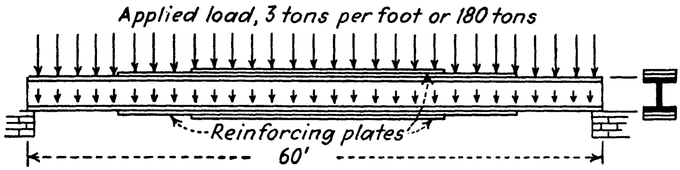


FIG. 388.

The maximum bending moment at mid span is

$$\frac{WL}{8} = \frac{wL^2}{8} = \frac{6300 \times 60^2}{8} = 2,835,000 \text{ ft.-lb.}$$

$$34,050,000 \text{ in.-lb.}$$

Hence the section modulus required at the center is

$$M = \frac{SI}{c} = SZ;$$

$$Z = \frac{34,050,000}{18,000} = 1890 \text{ in.}^3$$

The section modulus of the beam is only 1105 in.^{3} The beam must, therefore, be reinforced, and top and bottom cover plates will be used. Let the width of the cover plates be 16.65 in., the same as the width of the flanges of the I-beam.

If t is the required total thickness of the top (or bottom) cover plates at mid span,

$$\frac{20,290 + 2(16.65 \times t) \times \left(18.36 + \frac{t}{2}\right)^2}{18.36 + t} = 1890.$$

In solving by trial, the nearest commercial thickness is $t = 1.5$ in. Use two cover plates on top and two on the bottom, each $\frac{3}{4}$ in. thick.

The bending moment at a distance x from either pier is

$$M = \left(\frac{wLx}{2} - \frac{wx^2}{2}\right) 12 \text{ in.-lb.}$$

and the equation for the resisting moment just beyond the ends of the plates is

$$\left(\frac{6300 \times 60x}{2} - \frac{6300x^2}{2}\right)12 = 18{,}000 \times 1105.$$

Solving for x,

$$x = 10.7 \text{ ft. or } 49.3 \text{ ft.}$$

Accordingly, the first cover plates (the lower of the top plates, and the upper of the bottom plates) must have a minimum length of 38.6 ft.*

The section modulus of the beam reinforced by the first pair of cover plates is

$$\frac{20{,}290 + 2(16.65 \times 0.75) \times (18.36 + 0.375)^2}{18.36 + 0.75} = 1520 \text{ in.}^3$$

Now

$$\left(\frac{6300 \times 60x}{2} - \frac{6300x^2}{2}\right)12 = 18{,}000 \times 1520;$$

solving,

$$x = 16.8 \text{ ft. or } 43.2 \text{ ft.}$$

The length of the second pair of plates is 26.4 ft.*

Problem 227.—A 12-in.·50-lb. standard I-beam, 20 ft. long, is to be reinforced by cover plates 5 in. by $\frac{3}{8}$ in., one on the top and one on the bottom. What must be the minimum theoretical length of the cover plates so that the beam may carry the maximum allowable uniformly distributed load? Also find the load in addition to the weight of the beam. Take

$$S = 16{,}000 \text{ lb./sq. in.}$$

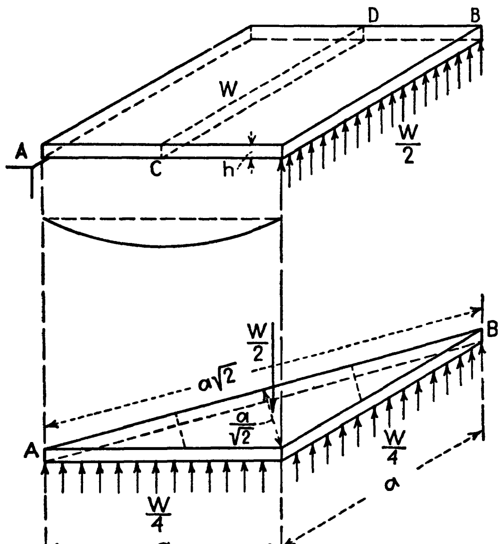
For the I-beam $I/c = 50.3$ in.^{3} and $I = 301.6.^4$

Ans. $W = 36{,}300$ lb.; length of plates 10.6 ft.

242. Flat Plate.—If a flat rectangular plate is supported at two opposite edges (Fig. 389) and is loaded (say) uniformly, the plate will bend as indicated in the figure. By assuming the plate of constant thickness h, cross-sections such as CD will remain rectangular during bending. The plate acts as a simple beam on end supports and the fiber stresses in the plate may be found by means of the flexure formula $M = SI/c$.

* *Note.*—The values of x computed above are theoretical maximum values. The lengths of the reinforcing plates must be increased to allow for the weakening effect of the rivet holes and to enable the rivets to transfer stress from the flanges of the beam into the reinforcing plates.

Assume now that the plate is supported at the four edges. The plate no longer acts as a simple beam on end supports. The corners of the plate tend to curl up and the supporting forces are not uniformly distributed along the edges. The flexure formula is not applicable. To show this, consider a square plate of thickness h. Take free the part of the plate lying to one



FIGS. 389 and 390.

side of the diagonal AB (Fig. 390). The moment of the external forces with respect to this diagonal is

$$M = 2 \times \frac{W}{4} \times \frac{a}{2\sqrt{2}} - \frac{W}{2} \times \frac{1}{3} \times \frac{a}{\sqrt{2}} = \frac{a}{\sqrt{2}} \times \frac{W}{12}.$$

If we assume the flexure formula applicable,

$$\frac{a}{\sqrt{2}} \times \frac{W}{12} = \frac{SI}{c} = S \times \frac{1}{6}a\sqrt{2} \times h^2$$

or

$$W = 4Sh^2. \tag{a}$$

Equation (a) would give the maximum safe load W for a given value of S if the moment along AB were uniformly distributed. In reality the moment along AB is not uniformly distributed and therefore Eq. (a) is an approximation. A mathematical analysis shows that the maximum safe load W for a homogeneous

square plate of constant thickness h is (taking Poisson's ratio as $m = 0.27$)

$$W = 3.15Sh^2.$$

243. Plate Uniformly Loaded.—The mathematical analysis for a flat plate supported all around its edge (or edges) is very complicated. Such a treatment is beyond the scope of this text. A few of the results, however, will be given for plates uniformly loaded.

W = load uniformly distributed over the plate.
S = maximum unit stress.
r = radius of circular plate.
a = side of square plate or
= longer side of a rectangular plate.
b = shorter side of a rectangular plate.
h = thickness of plate.
m = Poisson's ratio (Art. 22).

Circular plate, uniformly loaded.

Plate resting on a support all around its edge:

$$W = \frac{8\pi h^2 S}{3(3 + m)}.$$

Plate clamped all round its edge:

$$W = \frac{8\pi h^2 S}{3(1 + m)}.$$

Square plate, uniformly loaded.

Plate resting on a support all round its edges:

$$W = \frac{4h^2 S}{1 + m}.$$

Plate clamped all round its edges:

$$W = \frac{32h^2 S}{5(1 + m)}.$$

Rectangular plate, uniformly loaded, $a > b$.

Plate resting on a support all round its edges:

$$W = \frac{(a^2 + b^2)^2 h^2 S}{ab(a^2 + mb^2)}.$$

Plate clamped all round its edges:

$$W = \frac{4}{5}\,\frac{[3(a^2 + b^2)^2 - 4a^2b^2]h^2 S}{(a^2 + mb^2)ab}.$$

Problem 228.—The end of a steel water tank is a circular plate of radius $r = 8$ in. The water is under a pressure of 320 lb./sq. in. Assume plate clamped. Given $S = 16{,}000$ lb./sq. in. and $m = 0.27$, find the thickness of the plate. *Ans.* $h = 0.78$ in.

BEAMS OF TWO MATERIALS

244. Introduction.—Figure 391*a* shows a steel prism, and Fig. 391*b* a wooden prism. The two prisms are of equal length and carry equal axial loads. What must be the ratio of the area of the wooden prism to that of the steel prism if the two prisms are to elongate equally?

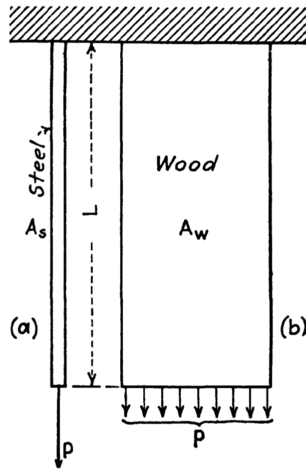


FIG. 391.

Let A_s = area of section of the steel prism.
A_w = area of section of the wooden prism.
E_s = the modulus of elasticity of steel.
E_w = the modulus of elasticity of wood.
n = the ratio E_s/E_w.
P = the axial load carried by each prism.
L = their common length.

Taking $E_s = 30{,}000{,}000$ lb./sq. in. and
$E_w = 1{,}500{,}000$ lb./sq. in., $\qquad n = 20.$

Since the prisms elongate equally [Art. 53, Eq. (1)],

$$\lambda = \frac{PL}{A_sE_s} = \frac{PL}{A_wE_w}.$$

Or

$$A_sE_s = A_wE_w.$$

Therefore

$$A_w = \frac{E_s}{E_w}A_s = nA_s. \tag{a}$$

Two prisms are said to be *equivalent* to each other if they are of the same length and elongate or contract equally under equal loads. From Eq. (*a*), it follows that, if a wooden prism is the equivalent of a steel prism, the area of the wooden prism (the wood equivalent) is n times that of the steel prism.

Since $P/A_s = S$ and $P/A_w = S_w$, we may write

$$\frac{SL}{E_s} = \frac{S_wL}{E_w}.$$

Or

$$S = \frac{E_s}{E_w}S_w = nS_w. \tag{b}$$

Hence the unit stress in the steel prism is n times the stress in the wood equivalent.

245. Beams of Wood and Steel.—The principles of the preceding article may be used to advantage when a beam composed of wood and steel is analyzed. Three cases will be considered.

Case I. *Wooden Beam Reinforced with Steel Plates Fastened to the Sides.* Fig. 392.—Replace the steel plates by their wood equivalents and determine the stresses in the resulting wooden beam (Fig. 392*b*). The stress at a point in the steel is n times that at the corresponding point in the wood equivalent.

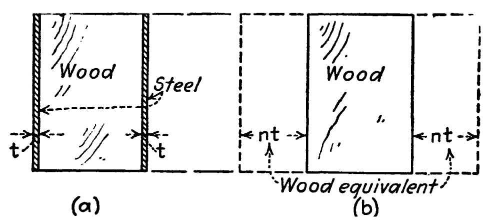


FIG. 392.

Note.—Cross-sections of the beam are assumed to remain plane sections during bending. Hence, for the wood and the steel to have corresponding fibers, the height of the wood equivalent should be the same as that of the steel, and the width of the wood equivalent (at a given distance from the neutral axis) must be n times that of the steel plate. That is, the prisms referred to in Art. 244 are now fibers of equal length at equal distances from the neutral axis.*

Example I. Fig. 392.—A wooden beam, 4 in. wide, 10 in. deep, and 16 ft. long, has a steel plate, $\frac{1}{4}$ by 10 in., fastened to each side. If the allowable stress in steel is S = 18,000 lb./sq. in. and that in wood is S_w = 1000 lb./sq. in., find the maximum uniformly distributed load the beam can carry. Take

$$n = E_s/E_w = 20.$$

* Fibers $\perp$ plane of paper.

Replace the steel plates by their wood equivalents. The resulting wooden beam (Fig. 392b) is then $4 + 2 \times 20 \times \frac{1}{4}$, or 14 in. wide. From Eq. (b), Art. 244, the allowable stress in the wood equivalent is $S/n = 900$ lb./sq. in. The stress in the steel governs since 900 is less than 1000.

$$M_m = \frac{WL}{8} = \frac{1}{8}W \times 16 \times 12 = 24W \text{ in.-lb.}$$

$$\frac{I}{c} = \frac{1}{6} \times 14 \times 10^2 = \frac{700}{3} \text{ in.}^3$$

Substituting in the flexure formula, we obtain

$$24W = 900 \times \frac{700}{3}.$$

Or

$$W = 8760 \text{ lb.}$$

Problem 229.—In the example above, assume that the allowable stress for wood is given as $S_w = 720$ lb./sq. in. *Ans.* $W = 7000$ lb.

Case II. *Wooden Beam Reinforced by Two Steel Plates, One on the Top and One on the Bottom.* Fig. 393.

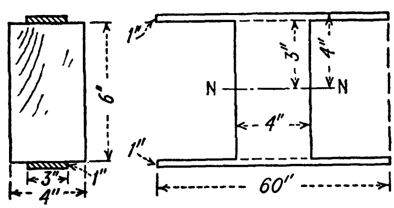


FIG. 393.

Example II. A wooden beam, 4 in. wide, 6 in. deep, and 12 ft. long, is reinforced by two steel plates one on the top and one on the bottom. Each plate is 1 in. thick and 3 in. wide. The beam carries a concentrated load of $P = 9000$ lb. at the middle of the span. Find the unit stress induced in the steel and the unit stress in the wood. Take $n = 20$.

If the steel plates are replaced by their wood equivalents, the resulting beam takes the form of an I-beam. The moment of inertia of the I-beam with respect to the neutral axis N is

For the flanges,

$$2\left(\frac{60 \times 1^3}{12} + 60 \times 3.5^2\right) = 1480$$

For the web,

$$\frac{4 \times 6^3}{12} = \underline{\quad 72}$$

$$I_N = 1552$$

$$M_m = \frac{9000 \times 12 \times 12}{4} = 324{,}000 \text{ in.-lb.}$$

$$= \frac{SI}{c}$$

$$= \frac{S \times 1552}{4}.$$

Then $S = 835$ lb./sq. in. equals the stress in the wood equivalent. Hence the stress in the steel is $20 \times 835 = 16{,}700$ lb./sq. in.

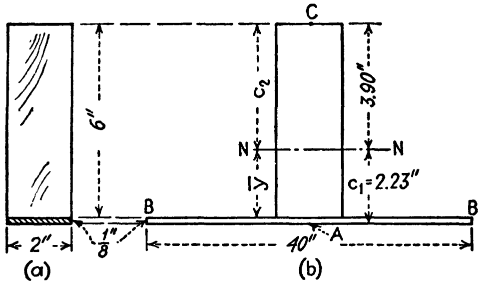


FIG. 394.

The maximum stress in the actual wooden part is $\frac{3}{4}$ of 835, or 626 lb./sq. in.

Problem 230.—A wooden beam, 4 in. wide and 6 in. deep, 20 ft. long, is reinforced by two steel plates, one at the top and one at the bottom. Each plate is 1 in. thick and 4 in. wide (see Example II above). The allowable stress for steel is $S = 18{,}000$ lb./sq. in. and that for wood is $S_w = 1000$ lb./sq. in. What uniformly distributed load can the beam safely carry?

Ans. 15,300 lb

Case III. *Wooden Beam Reinforced by a Steel Plate on the Bottom* (or on the Top). Fig. 394.

Example III.—A wooden beam, 2 in. wide, 6 in. deep, and 10 ft. long, is reinforced at the bottom by a steel plate 2 in. by $\frac{1}{8}$ in. If the allowable stress for steel is 18,000 lb./sq. in. and that for wood is 1200 lb./sq. in., find the maximum uniformly distributed load the beam can carry. Take $n = 20$.

Replace the steel plate by its wood equivalent (Fig. 394*b*).

Area of Section:

$$A = 2 \times 6 + 40 \times \tfrac{1}{8} = 17 \text{ sq. in.}$$

Position of the Neutral Axis N.—Take BB as an axis and use the formula $(\Sigma A)\bar{y} = \Sigma Ay$ [Eq. (7), Art. 137].

$$17 \times \bar{y} = 2 \times 6 \times 3 - 40 \times \tfrac{1}{8} \times \tfrac{1}{16} = 35.7;$$
$$\bar{y} = 2.10 \text{ in.}$$

Therefore

$$c_1 = 2.23 \text{ in.} \qquad \text{and} \qquad c_2 = 3.90 \text{ in.}$$

Moment of Inertia with Respect to Neutral Axis N.—Use the common line BB as a reference axis.

$$I_B = \tfrac{1}{3} \times 2 \times 6^3 + \tfrac{1}{3} \times 40 \times (\tfrac{1}{8})^3 = 144.$$

Then

$$I_N = I_B - A\bar{y}^2 = 144 - 17(2.10)^2 = 69.0 \text{ in.}^4$$

With 18,000 lb./sq. in. in the steel (outer fibers), the stress in the wood equivalent is 900 lb./sq. in. With 900 lb./sq. in. at A in the resulting wooden beam, the stress in the wood at C is

$$\frac{c_2}{c_1} \times 900 = \frac{3.90}{2.23} \times 900 = 1575 \text{ lb./sq. in.} > 1200.$$

The stress in the wood governs. Hence, at C,

$$\frac{1}{8}W \times 10 \times 12 = 1200 \times \frac{69}{3.90}.$$
$$W = 1420 \text{ lb.}$$

Problem 231.—Find the maximum stress in the steel and in the wood of the beam of Example III if a concentrated load of 700 lb. is applied at the middle.
Ans. 13,550 lb./sq. in. in steel and 1210 lb./sq. in. in wood.

246. Reinforced-concrete Beams.—Since concrete in tension is weak and unreliable, concrete beams are reinforced by imbedding steel rods or steel reinforcement in the concrete on the tension side of the beam. The steel is assumed to carry all of the tension. Figure 395 represents a section of a rectangular reinforced-concrete beam. The neutral axis is N. The part of the beam above N (the shaded part in section) is in compression.

Evidently, for an economical design, the ratio of steel to concrete should be such that simultaneously the full compressive stress of the concrete (above the neutral axis) and the full tensile stress of the steel (below the neutral axis) may be

utilized. For instance, if the compressive strength of concrete is 500 lb./sq. in. and the tensile strength of steel is 16,000 lb./sq. in., the beam should be so designed that when the unit compressive stress in the concrete reaches 500 lb./sq. in., the unit tensile stress in the steel will be 16,000 lb./sq. in.

A detailed discussion of reinforced-concrete beams is beyond the scope of this text. The design of such beams is complex. The investigation of a given beam, however, is comparatively a simple matter as the following example will show.

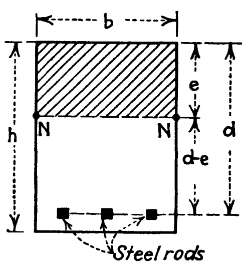


FIG. 395.

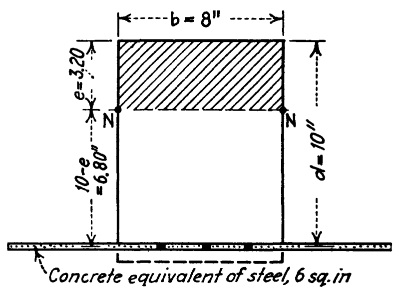


FIG. 396.

Example. Fig. 395. Using the standard notation for reinforced-concrete beams,

Let b = 8 in. = width of the concrete beam.

d = 10 in. = distance from compressive face to the plane of the steel.

A_s = 0.40 sq. in. = total area of the cross-section of the steel.

E_s = 30,000,000 lb./sq. in. = modulus of elasticity of steel.

E_c = 2,000,000 lb./sq. in. = modulus of elasticity of concrete.

$n = E_s/E_c = 15$.

f_s = unit tensile stress in the steel rods.

f_c = unit compressive stress in the concrete (upper fiber).

M_s = resisting moment as determined by the stress in the steel.

M_c = resisting moment as determined by the stress in the concrete.

It is required to find the maximum bending moment which this beam can carry if the safe stress in the steel is 16,000 lb./sq. in. and the safe stress in the concrete is 500 lb./sq. in.

Replace the steel by its concrete equivalent (Fig. 396). Since the area of the steel is A = 0.40 sq. in., the area of the concrete equivalent is $nA = 15 \times 0.40 = 6$ sq. in. [Eq. (a), Art. 244]. If the allowable stress in the steel is 16,000 lb./sq. in., then the

stress in its concrete equivalent is 16,000/15 = 1065 lb./sq. in. [Eq. (*b*), Art. 244]. To locate the neutral axis, take N as the axis of reference;

$$(8 \times e) \times \frac{e}{2} = 6(10 - e),$$

and $e = 3.20$ in. = distance of neutral axis from top face. Also, $10 - e = 6.80$ in. = distance of neutral axis from plane of steel.

The moment of inertia is

$$I_N = \frac{8 \times e^3}{3} + 6 \times (10 - e)^2 = 364.7 \text{ in.}^4$$

Hence (for the allowable stress in the concrete)

$$M_c = 500 \times \frac{364.7}{3.20} = 57{,}000 \text{ in.-lb.}$$

and (for the allowable stress in the concrete equivalent)

$$M_s = 1065 \times \frac{364.7}{6.80} = 57{,}000 \text{ in.-lb.}$$

Since $M_c = M_s$, the beam is of balanced design.

Problem 232.—In the beam of the example above, make the following changes: $b = 10$ in., $d = 10.58$ in., $A_s = 0.529$ sq. in. The beam is under a bending moment of $M = 80{,}000$ in.-lb. Find f_c and f_s.

Ans. $f_c = 500$ lb./sq. in.; $f_s = 16{,}000$ lb./sq. in.

CURVED BEAMS. HOOKS

247. Curved Beams.—Assume that a beam is subjected to a bending moment only. If λ equals the elongation or contraction of a fiber originally of length L, then, within the elastic limit [Eq. (2), Art. 53],

$$\lambda = \frac{SL}{E} \qquad \text{or} \qquad S = \frac{\lambda}{L}E = \frac{E}{L}\lambda. \tag{a}$$

If the beam is originally straight (Fig. 397*a*), all fibers originally are of the same length L. Hence, for a beam originally straight [Eq. (*a*)], S is directly proportional to λ and therefore to y, the distance of this fiber from the neutral axis. That is, within the elastic limit, the stress on a section of a beam originally straight is triangularly distributed (Fig. 397*b*). *Note* that the

flexure formula, $M = SI/c$, was derived on the assumption that the stress on a section of a beam is triangularly distributed.

Consider now a curved beam (Fig. 398*a*). Figure 398*b* represents the section of this beam as rectangular merely for con-

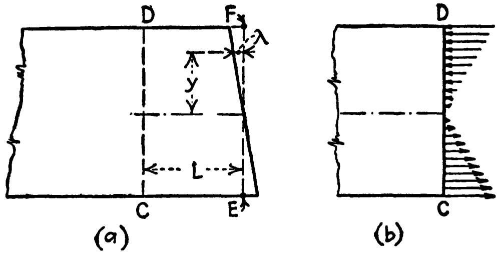


FIG. 397.

venience. It can be shown that when the curved beam is bent the neutral axis will lie above the centroidal axis as shown in the figure. Note that now the lengths of the fibers between two sections are not of the same length (Fig. 398*a*). Since both λ and L vary, S is not proportional to λ but to λ/L [Eq. (*a*)]. For a

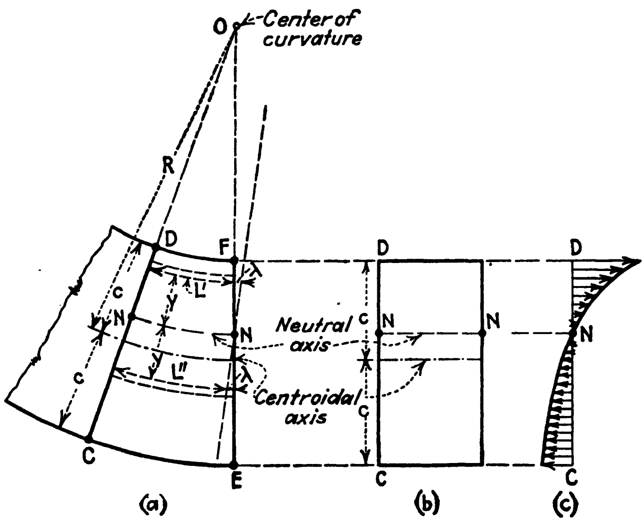


FIG. 398.

curved beam, therefore, S is not proportional to the distance of this fiber from the neutral axis. For instance, take two fibers equidistant from, but on opposite sides of, the neutral axis. Experiments warrant the assumption that, when a beam (straight

or curved) is bent, plane sections remain plane sections during bending. Hence we may assume that λ is numerically the same for two fibers equidistant from, but on opposite sides of, the neutral axis. If L' and L'' are, respectively, the lengths of these fibers (Fig. 398*a*), we may put [Eq. (*a*)]

$$\lambda = \frac{S'L'}{E} = \frac{S''L''}{E} \qquad \text{or} \qquad S' = \frac{L''}{L'}S''.$$

Since $L'' > L'$, $S' > S''$. That is, the stress in a fiber above the neutral axis is numerically greater than the stress in the corresponding fiber below the neutral axis. The distribution of the bending stress on the section CD is indicated in Fig. 398*c*. Since the stress is not triangularly distributed, the flexure formula is not applicable to curved beams.

248. Correction Factor.—A theoretically correct curved-beam formula may be derived.* In practice, however, it is more convenient to apply a correction factor to the formula for a straight beam. In following this practice, the stress in the outer fiber of a curved beam subjected to a bending moment M may be expressed as

$$S = K\frac{Mc}{I}$$

where Mc/I is the straight-beam formula and K is a correction factor determined from the correct curved-beam formula for each particular case.

If R equals the radius of curvature of the axis of the unloaded beam (Fig. 398*a*), and c equals the distance of the outer (inside) fiber from the centroidal axis, then for circular, elliptical, or rectangular sections K depends upon the value of R/c. For sections other than circular, elliptical, or rectangular, K usually depend also upon other dimensions of the section.

The following table gives values of K for

1. Circular or elliptical sections.
2. Rectangular sections.
3. Average values for other sections commonly used.

Note.—The fiber on the concave side of the beam, that is, the fiber DF (Fig. 398) will be called the *inside fiber;* and the fiber on the convex side (CE) will be called the *outside fiber*.

* See Boyd, "Strength of Materials," p. 350; also Maurer and Withey, p. 212; Poorman, p. 309; Seely, "Resistance of Materials," p. 336.

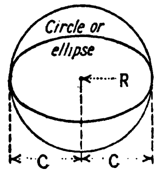


FIG. 399.

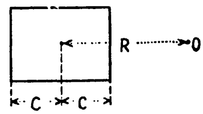


FIG. 400.

VALUES OF K FOR INSIDE AND OUTSIDE FIBERS

R/c	Circle or ellipse Fig. 399		Rectangle Fig. 400		Other sections, average values	
	Inside	Outside	Inside	Outside	Inside	Outside
1.2	3.41	0.54	2.89	0.57	When a section is unsymmetrical, R/c refers to the inside fiber	
1.4	2.40	0.60	2.13	0.63		
1.6	1.96	0.65	1.79	0.67		
1.8	1.75	0.68	1.63	0.70		
2	1.62	0.71	1.52	0.73	1.63	0.74
3	1.33	0.79	1.30	0.81	1.36	0.81
4	1.23	0.84	1.20	0.85	1.25	0.86
6	1.14	0.89	1.12	0.90	1.16	0.90
8	1.10	0.91	1.09	0.92	1.12	0.93
10	1.08	0.93	1.07	0.94	1.10	0.94
20	1.03	0.97	1.04	0.96	1.05	0.95

Note.—If $R/c > 20$ it is customary to use the straight beam formula with $K = 1$. Beams for which $R/c > 20$ are said to be *slightly curved.* If $R/c < 20$, the beam is said to be *sharply curved.* For a sharply curved beam the correction factor, as given in the table, should be used.

Example. Fig. 401.—The section of a curved beam is circular and its radius is $r = 1$ in. Hence $c = 1$. The radius of curvature of the axis of the bar is $R = 12$ in.

$$\frac{R}{c} = 12, \qquad K = 1.07.$$

Therefore

$$S = 1.07 \frac{Mc}{I} = 1.07 \frac{4M}{\pi \times 1^3}.$$

249. Hooks.—Figure 402 represents a hook carrying a load P. The maximum bending moment occurs at the section CD and is $M = Pe$ where e equals the distance of the line of action of P from the centroid of the section.

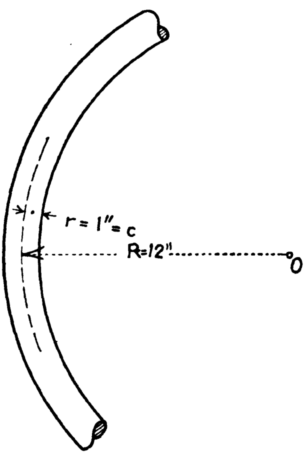


FIG. 401.

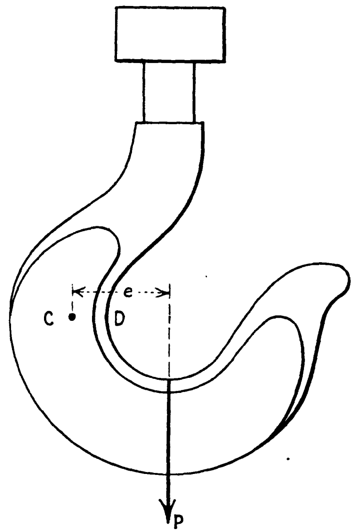


FIG. 402.

The stress on the section CD consists of (1) a direct stress whose intensity is

$$S_1 = \frac{P}{A},$$

and (2) a bending stress whose intensity in the outer fiber is

$$S_2 = K\frac{Mc}{I},$$

where K is a correction factor as explained in Art. 248. At D, the resulting stress is

$$S_D = S_1 + S_2 = \frac{P}{A} + K\frac{Pec}{I} \tag{a}$$

and, at C,

$$S_C = S_1 - S_2 = \frac{P}{A} - K\frac{Pec}{I}. \tag{b}$$

It should be remembered that the value of K for the inside fiber is not the same as that for the outside fiber (Table, Art. **248**).

Note.—Hooks usually are sharply curved so that R/c is less than 20. Hence it is seldom that K may be put equal to unity when analyzing hooks.

Example.—The section of a hook (Fig. 402) is a circle of radius $r = 1$ in. The radius of curvature is $R = 2$ in. $= e$. Find P if the allowable stress is 16,000 lb./sq. in.

For a circular section for the inside fiber, $K = 1.62$ if $R/c = 2$.

$$A = \pi r^2 = 3.14 \text{ sq. in.,} \qquad c = r = 1 \text{ in.,} \qquad I = \tfrac{1}{4}\pi r^4 = 0.785 \text{ in.}^4$$

Hence [Eq. (a)], for the inside fiber,

$$16{,}000 = \frac{P}{3.14} + 1.62\frac{P \times 2 \times 1}{0.785} \qquad \text{or} \qquad P = 3600 \text{ lb.}$$

With this value of P, the stress at C is (since for the outside fiber $K = 0.71$)

$$S_C = \frac{3600}{3.14} - 0.71 \times \frac{3600 \times 2 \times 1}{0.785} = -5360 \text{ lb./sq. in.}$$

(compression).

Note that

$$S_D = 16{,}000 \text{ lb./sq. in. (tension).}$$

PROBLEMS

233. A hook of rectangular section, 2 by 3 in., has a load applied 3 in. from the centroid of the section CD (Fig. 402). The radius of curvature of the axis of the hook is $R = 2.85$ in. What is the maximum load this hook can carry at 18,000 lb./sq. in.? *Ans.* $P = 10{,}300$ lb.

234. A wooden beam, 12 ft. long, rests on end supports. It has a constant width of $b = 4$ in. The beam is a double-truncated wedge symmetrical with respect to the center. The height of each end section is $h_1 = 2$ in. and the height of the section at the middle is $h_2 = 8$ in. What central load P can this beam carry if the safe fiber stress is 1200 lb./sq. in.? Also, when this load is applied, compute the stress in the outer fiber at a section 2 ft. from the middle of the span.

Ans. 1066 lb.; 1066 lb./sq. in.

235. A beam of constant width b and of length L is to rest on end supports and is to carry a uniformly distributed load $W = wL$. If S is the allowable fiber stress and v is the height of the section at a distance x from the end O, show that for a beam of uniform strength

$$v^2 = \frac{3w(L - x)x}{Sb}.$$

236. In Problem 235, if h is the required height of the beam at the center, show in full detail that

$$\frac{v^2}{h^2} = \frac{4(L - x)x}{L^2}.$$

237. A 3 by 12-in. wooden floor joist on end supports is 16 ft. long. It is desired to cut out a notch 2 in. square for the purpose of concealing a water pipe. How far from the end support may the inner edge of this notch be placed if the strength of the joist is not to be weakened? The load is uniformly distributed. *Ans.* $x = 3.57$ ft.

238. A 6-in. 14.75-lb. standard I-beam, 20 ft. long, is to be reinforced by steel plates, $\frac{1}{2}$ by 3 in., on the top and bottom flanges so that the beam may be able to carry a load of 480 lb./lin. ft. Determine the minimum theoretical lengths of the reinforcing plates if S is 18,000 lb./sq. in.

Ans. 14.2 ft. and 1.94 ft.

239. A steel plate is to be used to serve as a rectangular manhole cover in the upper lining of a concreted tunnel of rectangular section. The hole is 3 by 4 ft. and the plate must be able to support a depth of 6 ft. of earth. Assume the earth above the plate to weigh 7200 lb. Find the minimum safe thickness of the plate if S is 16,000 lb./sq. in. and $m = 0.30$.

Ans. $h = 0.402$ in.

240. A hook is to be made by bending a round steel rod. If R/c is not to be greater than 4, what size of rod should be used if the hook is to lift a load of 540 lb. acting through the center of curvature of the axis of the hook? Let S be 18,000 lb./sq. in. *Ans.* Diameter = 0.893 in.

CHAPTER XII

COLUMNS

250. Introduction.—Figure 403 represents a prism subjected to two equal, opposite, compressive forces Q and Q, one at each end.* For the present, assume that the forces act centrally upon the end sections. This is a restricted case. Consider the right section mn. Let F be the centroid of this section. If the prism does *not* bend (deflect), the line of action of Q and Q goes through F (Fig. 403a), and hence the stress on the section mn may be considered as uniformly distributed, its intensity being

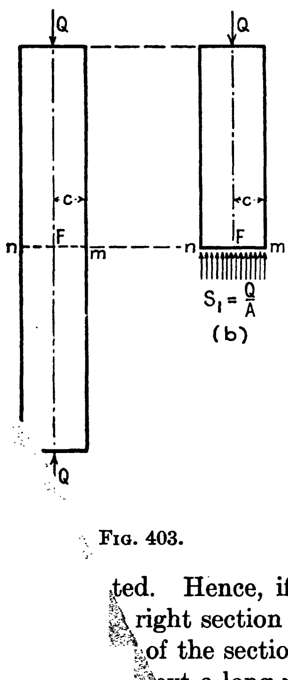


FIG. 403.

$$S_1 = \frac{Q}{A} \qquad (a)$$

where A = area of the section (see Art. 13).

If the length of a prism is less than eight to ten times its least lateral dimension, the prism is called a *short prism* or *block*. When such a prism is subjected to centrally applied compressive forces, the sidewise deflection of the prism is relatively so small that it may ted. Hence, if a short prism is centrally loaded, the right section is obtained by dividing the total load Q of the section [Eq. (a)].

xt a long prism, *i.e.*, a prism whose length is more n times its least lateral dimension (Fig. 404). Let bjected to *centrally applied end loads.* Assume value of Q the prism bends. Let a equal the rism at mid height. Since the line of action hrough the centroid of the section mn, the

member is called a *prism*, it is meant to signify nds it is a prism and therefore straight.

stress is not uniformly distributed. To determine the stress on the section nm, replace Q by an equal and parallel force Q_1 (Fig. 404b) whose line of action goes through F (the centroid of the section) and a couple whose moment is $M = Qa$. The force Q_1, as far as the section mn is concerned, will produce a direct compression whose intensity is (since $Q_1 = Q$)

$$S_1 = \frac{Q}{A}.$$

The moment Qa will produce a flexural stress. If, owing to this moment, S_2 equals the unit stress (compressive) in the outer fiber at m, then within the elastic limit [since $M = (S_2 I)/c$]

$$S_2 = \frac{Mc}{I} = \frac{Qac}{I}.$$

The maximum unit stress on the section mn is, therefore,

$$S = S_1 + S_2 = \frac{Q}{A} + \frac{Qac}{I}. \quad (1)$$

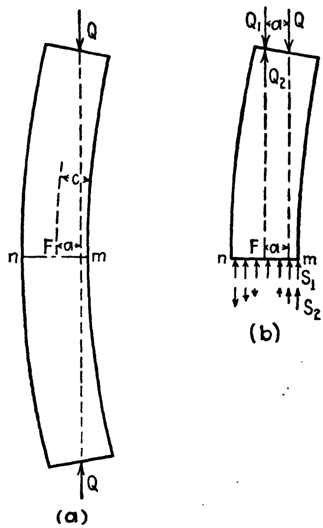


FIG. 404.

From Eq. (1) it is evident that the deflection of the prism has a decided influence upon the unit stress induced on a section of that prism.

251. Column Defined.—Any originally straight compression member is called a *column* if its deflection (lateral bending) must be considered when determining its strength. Pillars in buildings, compression members in bridges, piston rods and connecting rods in engines are examples of columns. Struts, posts, and braces are other terms for columns.

For the present, assume that the column is centrally loaded. Columns which are loaded eccentrically require special analysis (Art. 280). Centrally loaded columns may be divided into two classes on the basis of relative length as follows:

A *slender column* is one relatively so long that failure will be due primarily to flexure (bending). Consequently, S_1 may be neglected [Eq. (1)].

A *column of medium length* is one of such length that failure will be due to the combined effect of direct compression and flexure. Accordingly, both S_1 and S_2 must be considered [Eq. (1)].

Note.—A short prism or block under compression sometimes is called a *short column.* In a short column flexure may be neglected; *i.e.*, S_2 may be omitted in Eq. (1).

If the deflection of a column could be determined, the stress S in the outer fibers could be calculated. For instance (Fig. 404*a*), if the deflection a were known, S could be found from Eq. (1). There is, however, no theoretically correct expression that will determine a. Equation (1), therefore, cannot be used to find S. The analytical treatment of a column requires special procedure.

The purpose of this chapter is to derive equations that will give the relation between the breaking load (or else the working load) and the dimensions of the column. Such equations are called *column formulas.*

Note.—Columns need not be prismatic. In this chapter only prismatic columns will be considered.

252. An ideal column is one that is homogeneous, of constant cross-section, initially straight, and subjected only to centrally applied end loads. Actual columns never fully satisfy these conditions. Any slight variation from these conditions may have a marked effect upon the strength of a column. For instance, if a column has a slight initial crookedness, there will be an initial bending moment with loads centrally applied. This initial bending moment will produce a deflection. This deflection will increase the bending moment, which in turn will increase the deflection, and so on. The mutual dependence of bending moment and deflection is such that any deviation from ideal conditions may result in a decided reduction in the strength of the column.

The deviations of an actual column from ideal conditions are difficult and may even be impossible to include in a mathematical analysis. Hence, in deriving a column formula, an ideal column is first considered. The physical constants which theory introduces in a column formula are then determined experimentally so that the results obtained from the formula may be in fair agreement with actual conditions. Note that *the physical constants thus determined involve slight discrepancies from ideal conditions.* Sometimes the algebraic form of an equa-

tion for the strength of a column is modified to meet actual conditions. *Column formulas,* therefore, *are more or less empirical.*

Theoretically, an ideal column should not bend. There must be an initial bending moment to start the column to bend. An actual column, however, will always bend if subjected to a supposedly central loading. In the theoretical treatment of an ideal column it will be assumed that the column bends. It may be assumed, for instance, that the load is gradually

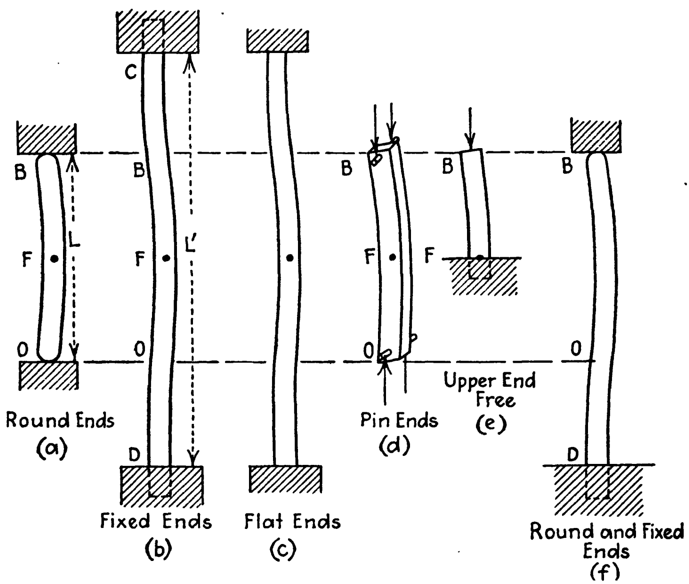


Fig. 405.

applied and that at the same time a lateral pressure is exerted against the column sufficient to cause the column to bend slightly, this lateral pressure being removed gradually as soon as the column can be held in a bent position by the end load.

253. Ideal End Conditions.—End conditions affect the strength of a column very much. In an actual column, the end conditions are difficult, if not impossible, to determine definitely. For ideal columns, end conditions may be classified as follows:

I. *Round Ends.* Fig. 405*a*.—Ends free to turn in all directions but not free to move laterally. As the ends turn, the load continues to act through the centroids of the end sections.

II. *Fixed Ends.* Fig. 405*b*.—Ends so rigidly held that the tangents to the column axis at the ends do not turn. In Fig. 405*b*, the ends are shown as built into unyielding abutments.

III. *Flat Ends.* Fig. 405*c*.—Ends faced to an accurate surface, perpendicular to the column axis, and bearing against plane surfaces that will not rotate.

IV. *Pin Ends.* Fig. 405*d*.—Ends free to turn in one direction but restrained in the other direction. For an ideal pin end, the pin is assumed smooth so as to eliminate friction and, furthermore, is held in an immovable bearing so that the end is fixed as regards flexure in the plane containing the axis of the pin.

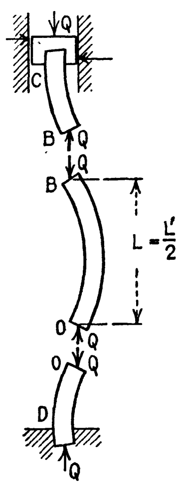


FIG. 406.

V. *End Free from Any Restraint.* Fig. 405*e*.

VI. *Combination of End Conditions.*—For instance (Fig. 405*f*), one end may be free to turn and the other end may be fixed.

254. Equivalent Lengths of Ideal Columns.—In Fig. 405*b*, the points O and B are points of inflection. At such a point there is *no bending moment.* If an ideal fixed-ended column is cut at the points of inflection O and B (Fig. 406) and central forces equal to Q are there applied, the three portions as separate columns will behave just as they do as parts of the original column DC. Now the portion OB is in the condition of a round-ended column. Evidently OB is one-half of DC.

That is, if

L' = length of the fixed ended column DC,

and

L = length of the equivalent round-ended column OB,

$$L = \frac{L'}{2}.$$

In like manner, in the column with one end fixed and the other end free to turn (Fig. 405*f*), the portion OB is in the condition of a round-ended column. A mathematical analysis shows that $OB = DB/\sqrt{2}$ (approximately).* Hence, if

* With reference to Fig. 405*b*, OB approximately equals $\frac{3}{4}DB$. The point B, however, does not lie on the vertical through D (Fig. 407*a*), but a little to the right of it. If then the column is cut at B and a central load Q is applied,

L' = length of the round- and fixed-ended column DB

and

L = length of the equivalent round-ended column OB,

$$L = \frac{L'}{\sqrt{2}}.$$

With reference to Fig. 405e, it is evident that if

L' = length of the column with one end free from restraint

and

L = length of the equivalent round-ended column,

$L = 2L'$.

Hence, given a column formula for an *ideal round-ended column* of length L, the corresponding column formulas for the other ideal end conditions just considered may be obtained by making the following substitutions:

For fixed ends (or flat ends), put $L/2$ for the L in the formula for round ends.

For round and fixed (*or flat*) *ends*, put $L/\sqrt{2}$ for the L in the formula for round ends.

For one end free from restraint, put $2L$ for the L in the formula for round ends.

Note.—It is evident, then, that in the theoretical treatment of columns round-ended columns are of primary interest.

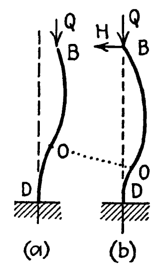


FIG. 407.

255. Euler's Column Formula for Round Ends.—An ideal slender column with round ends is made to bend (deflect) by an amount a (Fig. 408). The bent column is represented by its elastic curve OFB. It is required to find what central end load Q will hold the column in the bent position. Assume that the elastic limit is not exceeded.

Take O, the lower end of the elastic curve, as the origin, and the axis of the column in its unbent position as the x-axis. Take the y-axis as positive in the direction the column bends (to the left in Fig. 408), so that the y-coordinate of any point in the curve OFB is positive. Sometimes it is convenient to place the column

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a horizontal force H must be exerted at B to bring B directly over D (Fig. 407b). When this is done, the point of inflection O moves down slightly. That is, OB of Fig. 407a is not quite equal to OB of Fig. 407b. A mathematical analysis shows that OB (Fig. 407b) equals $0.7DB$ (very nearly), or approximately $DB/\sqrt{2}$. See "Strength of Materials," p. 288, by Maurer and Withey.

in the horizontal position as shown in Fig. 409. In this position the mathematical relations are more clearly seen.

From previous proof (Art. 176),

$$EI\frac{d^2y}{dx^2} = M$$

where M is the bending moment at the point N whose coordinates are x and y. It is necessary, however, to consider sign. If the bent column and the axes are taken as shown in Fig. 409, d^2y/dx^2 is intrinsically negative (Art. 177).* Hence M must be negative. Since y is positive, we must put $M = -Qy$. Therefore

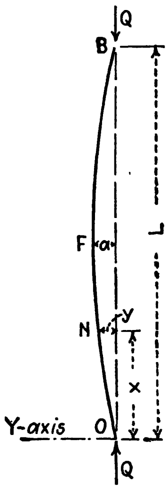


FIG. 408.

$$EI\frac{d^2y}{dx^2} = -Qy. \tag{a}$$

In Eq. (a), I is the moment of inertia of the section at N with respect to the centroidal axis (gravity axis) perpendicular to the plane of bending, *i.e.*, with respect to the centroidal axis perpendicular to the plane of the paper in Fig. 408 or Fig. 409.

Note that the sign given to M is in agreement with the rule of sign adopted in Art. 120. That is, if the axes are chosen as shown in Fig. 409, M is negative if the concavity is below the piece.

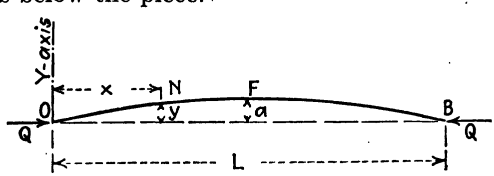


FIG. 409.

Equation (a) may be written

$$EI\frac{d}{dx}\left(\frac{dy}{dx}\right) = -Qy.$$

Or

$$EId\left(\frac{dy}{dx}\right) = -Qydx.$$

* That is, the derivative of a decreasing quantity is negative. Note that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ and that $\frac{dy}{dx}$ decreases as the point N moves to the right.

Multiplying through by dy/dx,

$$EI\frac{dy}{dx}d\left(\frac{dy}{dx}\right) = -Qydy.$$

Putting $u = dy/dx$,

$$EIudu = -Qydy.$$

Integrating,

$$EI\frac{u^2}{2} = -Q\frac{y^2}{2} + C_1$$

where C_1 = constant of integration.
Substituting dy/dx for u,

$$\frac{EI}{2}\left(\frac{dy}{dx}\right)^2 = -\frac{Qy^2}{2} + C_1. \tag{b}$$

This last equation must hold for any point in the elastic curve. It must hold, therefore, for the point F. At F, $dy/dx = 0$ and $y = a$.
Therefore

$$0 = -\frac{Qa^2}{2} + C_1 \qquad \text{or} \qquad C_1 = \frac{Qa^2}{2}.$$

Substituting this value of C_1 in Eq. (b), and simplifying,

$$EI\left(\frac{dy}{dx}\right)^2 = Q(a^2 - y^2).$$

Solving for dy/dx,

$$\frac{dy}{dx} = \pm\sqrt{\frac{Q}{EI}}\sqrt{a^2 - y^2}. \tag{c}$$

Using the plus sign and separating variables (see Art. 257),

$$\frac{dy}{\sqrt{a^2 - y^2}} = \sqrt{\frac{Q}{EI}}\,dx.$$

Integrating,

$$\sin^{-1}\frac{y}{a}\left(= \text{arc sin }\frac{y}{a}\right) = x\sqrt{\frac{Q}{EI}} + C_2 \tag{d}$$

where C_2 is the constant of integration. To determine C_2, substitute simultaneous known values of x and y. At O, $x = 0$ and $y = 0$.
Therefore

$$\sin^{-1} 0 = 0 + C_2.$$

Taking $\sin^{-1} 0 = 0$,

$$C_2 = 0$$

and Eq. (d) becomes

$$\sin^{-1} \frac{y}{a} = x\sqrt{\frac{Q}{EI}}. \tag{e}$$

This is the equation of the elastic curve and may be written

$$y = a \sin \left(x\sqrt{\frac{Q}{EI}}\right). \tag{f}$$

By assumption, the column is made to deflect an amount a, and it is required to find the value of Q that will maintain this deflection. That is, by assumption, $y = a$ when $x = L/2$. Substituting in Eq. (e),

$$\sin^{-1} \frac{a}{a} = \sin^{-1} 1 = \frac{L}{2}\sqrt{\frac{Q}{EI}}.$$

Or

$$\frac{\pi}{2} = \frac{L}{2}\sqrt{\frac{Q}{EI}}.$$

Solving for Q,

$$Q = \frac{\pi^2 EI}{L^2}. \tag{2}$$

Since $I = Ak^2$, where k is the radius of gyration [see Eq. (4), Art. 137],

$$\frac{Q}{A} = \frac{\pi^2 E}{\left(\frac{L}{k}\right)^2}. \tag{3}$$

Equation (2) or Eq. (3), is known as *Euler's formula* for a column with ends free to turn. The load Q is called the *Euler load.*

Equation (2) or Eq. (3) is seen to be independent of a, the deflection. That is, a load which will maintain a given deflection will maintain any deflection *so long as the elastic limit is not exceeded.* Hence a load slightly greater than the Euler load will bend the column until it fails. The Euler load therefore is the *critical load* and is often called the *breaking load for a slender column.*

Euler's formula was developed for an ideal column. In the derivation of this formula the direct stress $S_1 = Q/A$ was not considered. It was assumed that failure will be due solely to flexure. If the direct stress S_1 were appreciable, the column

would fail before the Euler load is reached. Hence *Euler's column formula is applicable to slender columns only.*

Actually, it is seldom that a column may be considered slender. Euler's column formula, however, is of great importance since it is used to determine practical column formulas as will be seen later. Moreover, in some cases, this formula enables one to obtain readily a first approximation to the necessary size for a column by solving for I.

Place one end of a long thin steel straightedge on a platform scale as shown in Fig. 410, and press down on the other end. The downward pressure may be determined by the ordinary process of weighing. After the straightedge begins to bend, it will be found that the deflection may be increased without any material change in the load Q. That is, within the elastic limit, Q is practically the same for all deflections. Moreover, it will be found that the value of Q thus determined experimentally will differ but slightly from the theoretical value determined by Euler's formula [Eq. (2)].

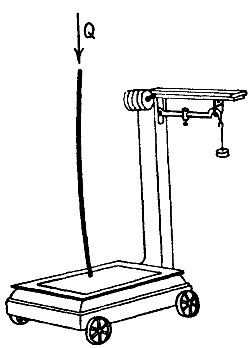


FIG. 410.

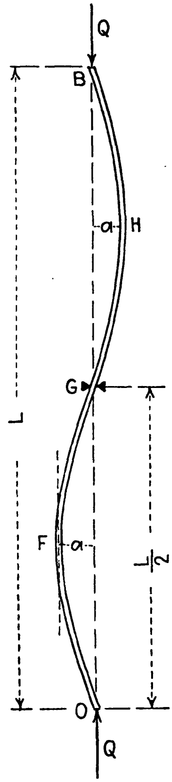


FIG. 411.

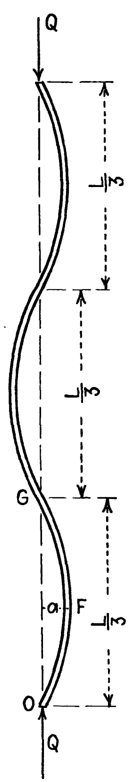


FIG. 412.

256. Note.—In the derivation of formula (2) or (3) it was assumed that the column bends as shown in Fig. 408. It is theoretically possible, however, for a column to bend in a number of ways. For instance, the column may be made to bend as shown in Fig. 411 if it is laterally supported at G as indicated in the figure. Each half of the column is in the condition of a round-ended column. Accordingly, in the column formula for round ends [Eq. (2)], put $L/2$ for L and obtain

$$Q = \frac{4\pi^2 EI}{L^2}.$$

That is, theoretically, a round-ended column of length L with a double bend is four times as strong as the same column with a single bend.

In like manner, if the column is made to bend as shown in Fig. 412,

$$Q = \frac{9\pi^2 EI}{L^2}.$$

These examples show the advantage of side bracing.

257. In dealing with a multivalued function, it is not always a simple matter to determine which value of the function should be used. The ultimate criterion is this: The equation or equations finally obtained must satisfy all assumptions made. For those interested in the mathematics of column problems, the following discussion may be of interest:

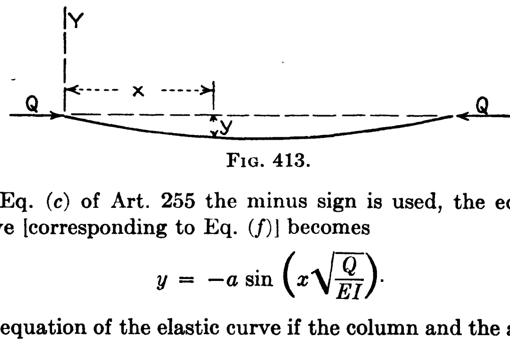


FIG. 413.

1. If in Eq. (*c*) of Art. 255 the minus sign is used, the equation of the elastic curve [corresponding to Eq. (*f*)] becomes

$$y = -a \sin\left(x\sqrt{\frac{Q}{EI}}\right).$$

This is the equation of the elastic curve if the column and the axes are taken as shown in Fig. 413.

2. Consider the round-ended column with two bends (Fig. 411). Performing the integration (as was done in Art. 255), we obtain

$$\sin^{-1}\frac{y}{a} = x\sqrt{\frac{Q}{EI}}. \qquad (e)$$

That is, in the derivation of Eq. (*e*), of Art. 255, no assumption was made that is not applicable to the column of Fig. 411 (or of Fig. 412). Let the student review the derivation of Eq. (*e*) to assure himself that this is the case. Referring now to Fig. 411, we have

At O,

$$x = 0, \qquad y = 0, \qquad \sin^{-1} 0 = 0.$$

(This condition was used to determine the constant of integration and therefore should not be used again.)

At F,

$$x = \frac{L}{4}, \qquad y = a, \qquad \sin^{-1} 1 = \frac{\pi}{2}.$$

At G,

$$x = \frac{L}{2}, \qquad y = 0, \qquad \sin^{-1} 0 = \pi.$$

At H,

$$x = \frac{3L}{4}, \qquad y = -a, \qquad \sin^{-1}(-1) = \frac{3\pi}{2}.$$

At B,

$$x = L, \qquad y = 0, \qquad \sin^{-1} 0 = 2\pi.$$

To determine the value of Q that will hold the column in the bent condition shown in Fig. 411, use any one of the foregoing conditions (except the first) and substitute the values in Eq. (*e*). For instance, putting $x = L/4$, $y = a$, and $\sin^{-1} 1 = \pi/2$,

$$\frac{\pi}{2} = \frac{L}{4}\sqrt{\frac{Q}{EI}} \qquad \text{or} \qquad Q = \frac{4\pi^2 EI}{L^2}.$$

258. Slenderness Ratio.—Euler's column formula for round-ended columns is [Eq. (3), Art. 255]

$$\frac{Q}{A} = \frac{\pi^2 E}{\left(\frac{L}{k}\right)^2} \tag{3}$$

where L = length of column.

k = radius of gyration of the cross-section with reference to that centroidal axis which is perpendicular to the assumed plane of bending, the *plane of bending* being defined as the plane of the elastic curve.

For instance, in Fig. 408, the column is represented by its elastic curve OFB, and this curve is assumed to lie in the plane of the paper. Hence the plane of the paper is the assumed plane of bending, and k is the radius of gyration of the cross-section with respect to the centroidal axis g-g, which is perpendicular to the plane of the paper.

The ratio L/k is called the *slenderness ratio*. The slenderness ratio enters into all column formulas. It is important, therefore, to be able to determine the k that should be used in a particular case. If a round-ended column, centrally loaded, is free to bend in any direction, the column will bend in the plane of the least radius of gyration. Hence, if a centrally loaded column is free to bend in any direction, the value of k to be used is the least radius of gyration. From Eq. (3), the load Q will have the least value when L/k is a maximum, *i.e.*, when k is the least radius of gyration.

Wooden columns usually are rectangular, and commercial column formulas for wooden columns often are expressed in terms of L/d where d is the dimension of the cross-section parallel to the plane of bending. If the plane of the paper is the assumed plane of bending, then d is the dimension (of section) parallel to the plane of the paper.

Note.—Unless a statement to the contrary is made, it will be assumed that the column is free to bend in any direction.

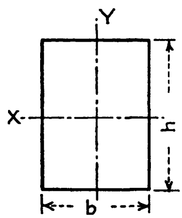


FIG. 414.

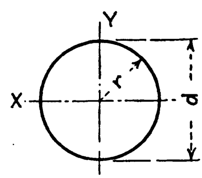


FIG. 415.

259. Radius of Gyration.—*Rectangular section* of dimensions b and h (Fig. 414):

$$I_x = \frac{bh^3}{12} = (bh)\frac{h^2}{12} = Ak_x^2.$$

Therefore

$$k_x^2 = \frac{h^2}{12} \quad \text{and} \quad k_x = \frac{h}{\sqrt{12}}.$$

$$I_y = \frac{hb^3}{12} = (bh)\frac{b^2}{12} = Ak_y^2.$$

Therefore

$$k_y^2 = \frac{b^2}{12} \quad \text{and} \quad k_y = \frac{b}{\sqrt{12}}.$$

Hence, if d is the smaller of the two, b or h,

$$\text{Min. } k = \frac{d}{\sqrt{12}} = \frac{d}{2\sqrt{3}}.$$

Solid Circular Section. Fig. 415.

$$I_x = I_y = \frac{\pi r^4}{4} = (\pi r^2)\frac{r^2}{4} = Ak^2.$$

Therefore

$$k^2 = \frac{r^2}{4} \quad \text{and} \quad k = \frac{r}{2} = \frac{d}{4}.$$

Here d is the diameter of the circular section.

Hollow Circular Section. (Fig. 416).—Outer radius equals $r = d/2$; inner radius equals $r_1 = d_1/2$.

$$I_x = I_y = \frac{\pi r^4}{4} - \frac{\pi r_1^4}{4} = \pi(r^2 - r_1^2)\frac{r^2 + r_1^2}{4} = Ak^2.$$

Therefore

$$k = \frac{\sqrt{r^2 + r_1^2}}{2} = \frac{\sqrt{d^2 + d_1^2}}{4}.$$

The values of k for other sections may be found in the many handbooks for engineers.

Example I.—A hollow round column has an outer diameter of $d = 8$ in. and an inner diameter of $d_1 = 6$ in. Find its radius of gyration.

$$k = \frac{\sqrt{d^2 + d_1^2}}{4} = \frac{\sqrt{8^2 + 6^2}}{4} = 2.50 \text{ in.}$$

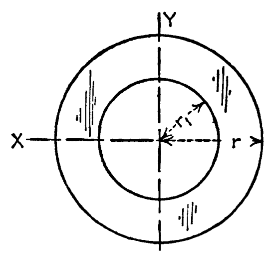


FIG. 416.

Example II.—Find the radius of gyration of a solid round column of the same sectional area as that of Example I. Let d equal the diameter of the solid section.

$$A = \frac{\pi d^2}{4} = \frac{\pi(8^2 - 6^2)}{4}.$$

Therefore

$$d = \sqrt{28} = 5.29 \text{ in.}$$
$$k = \frac{d}{4} = \frac{5.29}{4} = 1.323 \text{ in.}$$

Problem 241.—In a triangle, b is the base and h is the altitude. Let X be a gravity axis parallel to the base. Given $I_x = \frac{1}{36}bh^3$, show that

$$k_x = \frac{h}{\sqrt{18}}.$$

260. Effect of Direct Stress.—If Euler's formula is plotted by taking L/k as abscissa and Q/A as ordinate, the curve thus obtained is called *Euler's curve.* In Fig. 417, E_r is Euler's curve for a round-ended mild-steel column with $\pi^2 E = 300{,}000{,}000$. The curve E_r represents the equation

$$\frac{Q}{A} = \frac{\pi^2 E}{\left(\frac{L}{k}\right)^2} = \frac{300{,}000{,}000}{\left(\frac{L}{k}\right)^2}. \qquad (a)$$

Since $\pi^2 = 9.87$ . . . , this is equivalent to taking

$$E = 30{,}400{,}000 \text{ lb./sq. in.}$$

The curved line T represents the mean of the results of Tetmajer's tests on round-ended mild-steel columns. The curve T is an experimental curve based upon the average values of the results obtained from carefully made tests on round-ended mild-steel columns. Note that for $L/k > 120$, the two curves T and E_r practically coincide, and that for $L/k < 120$ the two curves diverge more and more as L/k decreases. This means that for L/k less than 120 the direct stress Q/A cannot be neglected [see Eq. (1), Art. 250], and that the effect of the direct

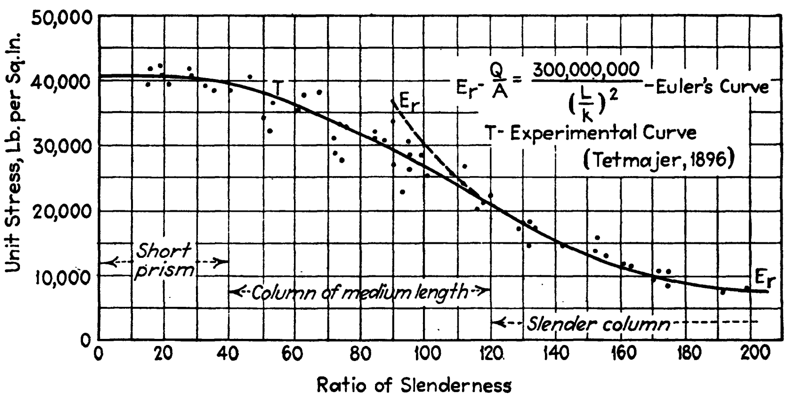


FIG. 417.

stress becomes more and more pronounced as L/k decreases. Note also that for $L/k < 40$ the experimental curve is practically a straight horizontal line. This indicates that if $L/k < 40$ failure is due almost wholly to the direct stress. Hence

If $L/k > 120$, the column is a slender column.

If $40 < L/k < 120$, the column is of medium length.

If $L/k < 40$, the member may be treated as a short prism or block and no column formula is needed.

It is accordingly seen that if L/k is less than 120 Euler's column formula should not be used for round-ended mild-steel columns. For columns in general it is not possible to set a value for L/k below which Euler's formula should not be used, since the value of Q depends also upon the kind of material in the column and upon the "end conditions."

Example.—In a laboratory test, a solid mild-steel rod is used as a round-ended column. The radius of the rod is $r = \frac{1}{2}$ in. and the length is $L = 3$ ft.

$$k = \frac{r}{2} = 0.25 \text{ in. (Art. 259.)}$$

$$\frac{L}{k} = \frac{36}{0.25} = 144. \qquad A = \pi r^2 = 0.785 \text{ sq. in.}$$

The column is a slender column ($L/k > 120$), and from Eq. (a),

$$\frac{Q}{0.785} = \frac{300{,}000{,}000}{144^2}.$$

Therefore

$$Q = 11{,}380 \text{ lb.}$$

Problem 242.—A 1- by 3-in. rectangular steel rod, 3 ft. in length, is used as a round-ended column. Find the breaking load.

Ans. $L/k = 125$; $Q = 57{,}600$ lb.

261. Rankine's Formula.—Figure 418 represents a round-ended column of medium length carrying a central load Q. The maximum unit stress S occurs in the outer fibers at m.

From Art. 250 [Eq. (1)],

$$S = S_1 + S_2 = \frac{Q}{A} + \frac{Qac}{I}$$

where a = the deflection of the column.

Substituting Ak^2 for I,

$$S = \frac{Q}{A} + \frac{Qac}{Ak^2} = \frac{Q}{A}\left(1 + \frac{ac}{k^2}\right). \qquad (a)$$

Or

$$\frac{Q}{A} = \frac{S}{1 + \dfrac{ac}{k^2}}. \qquad (b)$$

Note that, whereas S is the *maximum unit stress*, Q/A is the *average unit stress on the section mn.*

Within the elastic limit, Eq. (a) is theoretically correct. This equation contains a, the deflection. However, no theoretically correct expression for a can be found and it is necessary to resort to approximations.

Let Q equal the breaking load; *i.e.*, Q equals the minimum load that will cause the column to fail. Note that at failure the elastic

limit is exceeded and that therefore Eq. (*b*) does not strictly apply. Note also that a slight deviation from the assumed ideal conditions may have a marked effect upon the strength of the column. Accordingly, the value to be given S in Eq. (*b*) is in doubt. Hence put $S = C$, a constant (different for different materials) to be determined experimentally. Moreover, experience warrants the assumption that if we put $ac = qL^2$, where q is another constant to be determined by experiment, results will

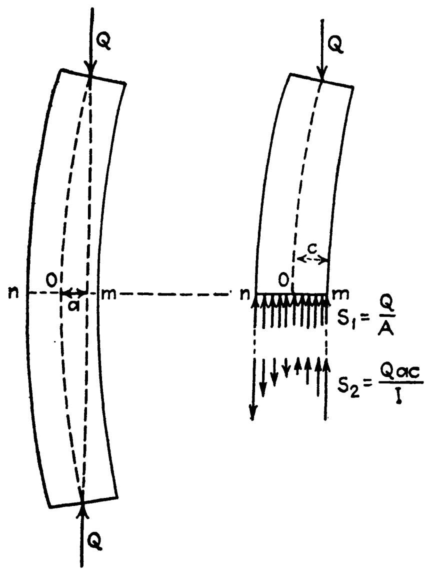


FIG. 418.

be obtained which are sufficiently accurate for practical purposes. Making these substitutions, Eq. (*b*) becomes

$$\frac{Q}{A} = \frac{C}{1 + q\left(\frac{L}{k}\right)^2}. \tag{4}$$

As stated above, C and q are experimental constants. L/k is the slenderness ratio of the column and Q is the breaking load.

This formula is known as *Rankine's formula* and the curve representing the formula is called *Rankine's curve.*

Illustration.—For round-ended mild-steel columns, Rankine and other authors give $C = 50{,}000$ lb./sq. in. and $q = 1/9000$.

Hence

$$\frac{Q}{A} = \frac{50{,}000}{1 + \frac{1}{9000}\left(\frac{L}{k}\right)^2}, \qquad \text{and} \qquad \left(\frac{L}{k} > 50\right). \tag{5}$$

This is Rankine's formula for breaking loads for mild-steel columns under round-ended conditions.

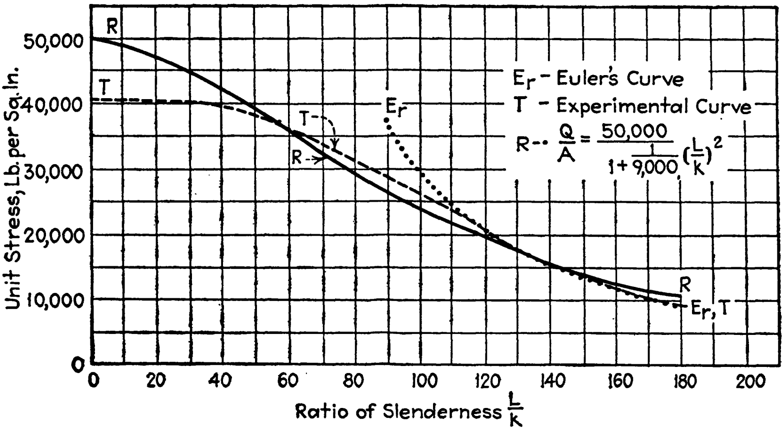


FIG. 419.

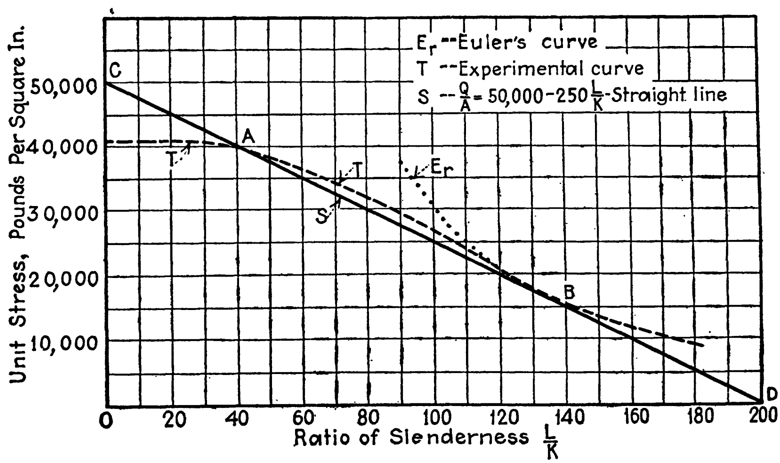


FIG. 420.

In Fig. 419, R is Rankine's curve [Eq. (5)], E_r is Euler's curve, and T is the experimental curve based upon results obtained from carefully made tests on round-ended mild-steel columns. Note that for $L/k < 50$ Rankine's curve gives values for Q/A that

are too large. If $L/k < 50$, no column formula is needed. If $L/k > 50$, Rankine's curve is in fair agreement with the experimental curve T.

Example.—A mild-steel rod of length $L = 2$ ft. and radius $r = \frac{1}{2}$ in. is used as a round-ended column in a carefully made laboratory test. For what value of Q may this column be expected to fail? $k = \dfrac{1}{4}$ in., $\dfrac{L}{k} = 96$, $A = 0.785$ sq. in.

$$\frac{Q}{0.785} = \frac{50,000}{1 + \frac{1}{9000} \times 96^2} = 24,700.$$

Therefore

$$Q = 19,400 \text{ lb.}$$

262. Straight Line Formula.—Referring to Fig. 420, note that between A and B the experimental curve T is approximately a straight line. This suggests the *straight line formula.* Draw the straight line CD. This line agrees fairly well with the experimental curve between A and B. Note that this line is drawn through the point C where $Q/A = 50,000$ and the point D where $L/k = 200$. Observe also that the line CD is very nearly tangent to Euler's curve at B.

To find the equation of the straight line CD, let $x = L/k$ and $y = Q/A$. The equation of the straight line is

$$y = 50,000 + mx$$

where m is the slope of the line and equals $-50,000/200 = -250$. Therefore

$$\frac{Q}{A} = 50,000 - 250\left(\frac{L}{k}\right) \text{ (breaking load, mild steel, round-ended)} \qquad (6)$$

provided L/k lies between 40 and 140. If L/k is less than 40 no column formula is needed. If L/k is greater than 140, Euler's formula may be used.

There is a straight line formula for each type of column and for each kind of material.

263. In Fig. 420 the line CD is drawn very nearly tangent to Euler's curve at B. Note that, at B, $Q/A = 50,000/3$ (very nearly). In general, let the straight line formula take the form

$$\frac{Q}{A} = C - b\frac{L}{k}$$

where C and b are constants. It can be shown that if the straight line represented by this equation *is tangent to Euler's curve*, then at the point of tangency $Q/A = C/3$.

In some cases, wooden columns for instance (see Fig. 424), the line CD is drawn to cut Euler's curve. If the line CD were drawn tangent to Euler's curve, the value of Q/A as given by the straight line formula for values of L/k between A and B would be considerably less than the Q/A from the experimental curve.

264. Parabolic Formula for Columns.*—In some cases, the *parabolic formula*

$$\frac{Q}{A} = C - b\left(\frac{L}{k}\right)^2 \tag{7}$$

gives results more closely in agreement with experimental data than either Rankine's formula or the straight line formula.

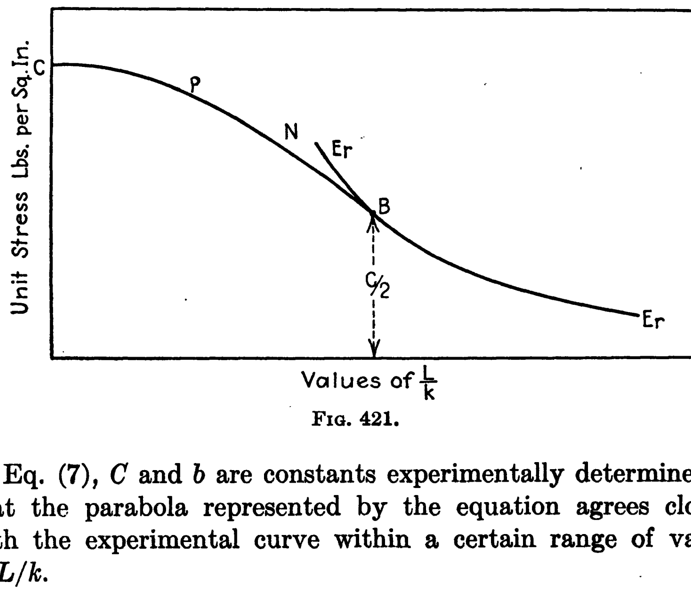


FIG. 421.

In Eq. (7), C and b are constants experimentally determined so that the parabola represented by the equation agrees closely with the experimental curve within a certain range of values of L/k.

In Fig. 421, E_r is Euler's curve and P is the parabolic curve. It can be shown that, if the constants C and b of Eq. (7) are determined so that P is tangent to E_r at B, then at B, $Q/A = C/2$.

265. Factor of Safety. Safe or Working Load.—In the foregoing column formulas, Q is the load under which the column

* The column formulas recently adopted by the American Railway Engineering Association are parabolic formulas. See Art. 270.

may be expected to fail. In other words, Q is the breaking load.

If n is the factor of safety to be used, then the safe or working load is $P = Q/n$ or $Q = nP$.

With a factor of safety of $n = 2.5$, Eq. (6) becomes

$$\frac{P}{A} = 20{,}000 - 100\frac{L}{k} \text{ (safe load, mild steel, round-ended conditions, } n = 2.5). \quad (8)$$

If L/k is less than 40, no column formula is needed. By putting $L/k = 40$, $P/A = 16{,}000$ lb./sq. in. That is, if L/k is less than 40, the member is designed as a short prism with an allowable unit stress of 16,000 lb./sq. in.

Example I.—A 12-in. 28-lb. WF beam, 10 ft. long, is to be used as a round-ended column. Using a factor of safety of 2.5, find the safe load the column can carry.

The least radius of gyration is $k = 1.46$ in. Hence

$$\frac{L}{k} = \frac{120}{1.46} = 82.2.$$

Also, $A = 8.23$ sq. in. Using formula (8),

$$\frac{P}{A} = 20{,}000 - 100 \times 82.2 = 11{,}780 \text{ lb./sq. in.}$$

Therefore

$$P = 97{,}000 \text{ lb.}$$

Example II.—Assume that the column of Example I is laterally supported so that it will bend in a plane perpendicular to axis 1–1. Then $k = 5.09$ in. and $L/k = 23.6$. Since L/k is less than 40, the member is designed as a short prism with an allowable stress of 16,000 lb./sq. in.

So

$$P = 16{,}000 \times 8.23 = 131{,}700 \text{ lb.}$$

COMMERCIAL COLUMNS

266. In commercial columns (columns in buildings, bridges, etc., *i.e.*, columns used in engineering structures), the deviations from ideal conditions may be quite marked, so much so that practical column formulas are only rough approximations at best. Such formulas necessarily must be conservative since they must make allowance for such possible deviations from ideal conditions as these: defects and lack of homogeneity in the mate-

rial, variations in the cross-section of the column, an accidental eccentricity in the loading, an initial crookedness in the column. It should be kept in mind that *laboratory tests on selected columns that satisfy ideal conditions as nearly as possible may give results quite at variance with the results obtained from commercial column formulas.* In the following pages, commercial or practical column formulas will be considered.

267. Pin-ended Columns.—Round-ended columns are not used in practice. Pin-ended columns are used but not to the same extent as formerly. It is seldom that a column in practice is fixed or flat ended in the same way that a test-piece in a testing machine may be fixed or flat ended. For instance, columns in bridges, buildings, etc., usually are riveted to other members of the structure and at first sight may seem to be fixed ended. They are not fixed ended, however, since the members to which they are riveted are not fixed or rigid. Moreover, tests on such columns show that they are little better, as far as strength is concerned, than pin-ended columns. It has become, therefore, a general practice among engineers *to treat commercial columns as pin ended.**

Occasionally, however, it is known that the ends of the column are well restrained and that the load is very nearly centrally applied. In such cases, a formula for fixed ends may be used. As stated above, this is seldom the case in practice.

268. Euler's Formulas Modified to Meet Actual End Conditions.—Theoretically a round-ended column and a pin-ended column are essentially alike. Practically, however, they are not alike. Friction between pin and column brings in an end restraint that in a particular case is difficult if not impossible to determine. Experiments show that a column with pin ends is stronger than the same column with round ends. Experiments also show that Euler's formula for fixed or flat ends must be modified to meet actual end conditions.

The following formulas give results fairly consistent with the results of tests on *slender* columns.

Round ends:

$$\frac{Q}{A} = \frac{\pi^2 E}{\left(\frac{L}{k}\right)^2} = \frac{10E}{\left(\frac{L}{k}\right)^2} \text{ (approximately).}$$

* In the 1935 Specifications for Railway Bridges a distinction is made between pin-ended and riveted-ended columns.

Pin ends:

$$\frac{Q}{A} = \frac{16E}{\left(\frac{L}{k}\right)^2}. \tag{9}$$

Fixed or flat ends:

$$\frac{Q}{A} = \frac{25E}{\left(\frac{L}{k}\right)^2}. \tag{10}$$

Note.—For an ideal column, the coefficient in Eq. (10) should be $4\pi^2$.

269. Given two columns of the same cross-section and of the same kind of material so that A, k, and E are the same for both columns. Let one column of length L_p be pin ended and the other of length L_f be fixed or flat ended. Assume that the two columns are equally strong. Equating the right-hand members of Eqs. (9) and (10),

$$\frac{16}{L_p^2} = \frac{25}{L_f^2} \qquad \text{or} \qquad L_p = \frac{4}{5}L_f.$$

Hence, *if for the L in a practical column formula for pin ends we put $\frac{4}{5}L$, the corresponding practical column formula for fixed or flat ends is obtained.*

270. Structural-steel Columns.—In 1909, a committee on steel columns was appointed by the American Society of Civil Engineers (A. S. C. E.). In 1912, a similar committee was appointed by the American Railway Engineering Association (A. R. E. A.). A careful study of all the available data on the subject led both committees to recommend for structural steel the following formula for the breaking load:

$$\left.\begin{aligned} \frac{Q}{A} &= 37{,}500 - 125\frac{L}{k}; \\ 50 &< \frac{L}{k} < 225. \end{aligned}\right\} \text{Pin ends, structural steel, breaking load} \tag{11}$$

In Fig. 422, E_p represents Euler's modified formula for pin ends. The modulus of elasticity is taken as

$$E = 30{,}000{,}000 \text{ lb./sq. in.}$$

The straight line S represents Eq. (11). The curved line R represents Rankine's formula

$$\left.\begin{aligned}\frac{Q}{A} &= \frac{37{,}500}{1 + \dfrac{1}{18{,}000}\left(\dfrac{L}{k}\right)^2};\\ \frac{L}{k} &> 60.\end{aligned}\right\}\text{Pin ends, structural steel, breaking load} \qquad (12)$$

Note that the straight line S is very nearly tangent to Euler's curve E_p, and that Rankine's curve R gives fairly satisfactory results if $L/k > 60$.

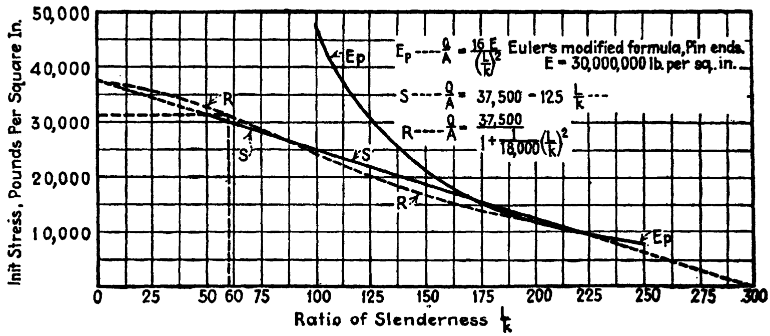


FIG. 422.

By using a factor of safety of $n = 2.08$ (as is done in designing buildings, etc.),

$$\left.\begin{aligned}\frac{P}{A} &= \frac{18{,}000}{1 + \dfrac{1}{18{,}000}\left(\dfrac{L}{k}\right)^2}\\ \frac{L}{k} &> 120\end{aligned}\right\} \quad \text{(A.I.S.C.)} \qquad (13)$$

In 1936 formula (13) was adopted by the American Institute of Steel Construction (A.I.S.C.). And, for L/k not greater than 120, they adopted the formula

$$\left.\begin{aligned}\frac{P}{A} &= 17{,}000 - 0.485\left(\frac{L}{k}\right)^2\\ \frac{L}{k} &\leq 120\end{aligned}\right\} \quad \text{(A.I.S.C.)} \qquad (14)$$

Formula (14) is a column formula of the parabolic type.

In 1935 the American Railway Engineering Association adopted the following parabolic formulas:

$$\frac{P}{A} = 15{,}000 - \frac{1}{4}\left(\frac{L}{k}\right)^2 \quad \text{Riveted ends} \quad (15)$$

$$\frac{L}{k} \leq 140$$

$$\frac{P}{A} = 15{,}000 - \frac{1}{3}\left(\frac{L}{k}\right)^2 \quad \text{Pin ends} \quad (16)$$

The committee of the A.R.E.A. which made the above recommendation used the secant formula (Art. 281) as a basis.

Note.—If the column is made of some ductile material such as structural steel, a load that will bring the unit stress in the outer fiber up to the yield-point will, if continued, cause the column to fail. This seems evident from the fact that as soon as the yield-point is reached, the outer fibers at m (Fig. 418) will suddenly shorten (yield) without any appreciable increase in the unit stress in these fibers. This will result in an increased deflection and therefore in an increase in the bending moment Pa without a corresponding increase in the unit stress at m. Hence the column will fail. The factor of safety to be used when a column is made of some ductile material should be based, therefore, upon the yield-point. The yield-point for ductile steel is about one-half of the ultimate so that a factor of safety of 2 or 2.5 based upon the yield-point is equivalent to a factor of safety of 4 or 5 based upon the ultimate stress.

In general, a column is apt to fail as soon as the elastic limit is exceeded to any appreciable extent. This explains why in the case of a column the factor of safety as a rule is lower than that used for the same material in direct compression or tension.

271. Cast-iron Columns.—Formerly cast-iron columns were extensively used. At present, steel columns are preferred.

Working formula recommended for cast-iron columns:

$$\frac{P}{A} = 9000 - 40\frac{L}{k} \text{(N. Y. law; } n = \text{about 4)} \quad (17)$$

$$\frac{L}{k} < 125.$$

Or

$$\frac{P}{A} = \frac{8500}{1 + \frac{1}{12{,}500}\left(\frac{L}{k}\right)^2}; \quad (18)$$

$$\frac{L}{k} < 125.$$

The constants in formula (18) were selected to make the two curves in fair agreement for $L/k < 125$, as is shown in Fig. 423.

Note.—The Building Code Committee of the U.S. Department of Commerce recommended that formula (17) be used for values of L/k up to 90.

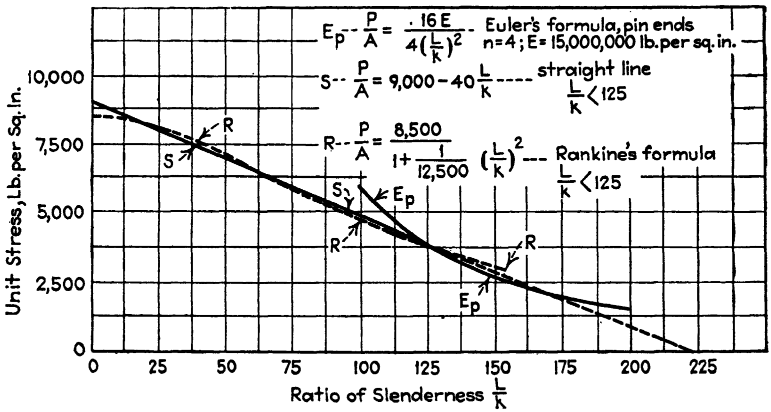


FIG. 423.

Example.—A round, hollow cast-iron column is 12 ft. long. Its outer radius is $r = 4$ in. and its inner radius is $r_1 = 3$ in. Find the safe load.

$$A = \pi(4^2 - 3^2) = 22 \text{ sq. in.};$$

$$k = \frac{\sqrt{3^2 + 4^2}}{2} = 2.50 \text{ (see Art. 259)}.$$

$$\frac{L}{k} = \frac{144}{2.5} = 57.6.$$

$$\frac{P}{22} = 9000 - 40 \times 57.6.$$

Therefore

$$P = 147{,}500 \text{ lb.}$$

272. Wooden Columns.—There are several kinds of wood used for columns. Each kind is divided into two main divisions, *common* and *select*. A rational column formula for wood should consider, therefore, not only the kind of wood used but also the grade. Now a wood of a particular kind and grade varies appreciably in its physical properties, so much so that a column formula for wood must be very conservative.

The column formula that seems to have the preference is that developed by the Forest Products Laboratory (F.P.L.) of the Federal Government. The formula is

$$\frac{P}{A} = S\left[1 - \frac{1}{3}\left(\frac{L}{Kd}\right)^4\right]; \qquad \frac{P}{A} \equiv\!\!> \frac{2}{3}S. \tag{19}$$

where S = allowable compressive stress for a block of the given kind and grade of wood.

d = least lateral dimension (rectangular section).

K = constant defined as

$$K = \frac{\pi}{2}\sqrt{\frac{E}{6S}}. \tag{a}$$

By the use of tables, formula (19) is readily applied.

Figure 424 is constructed for select hemlock.

$$E = 1{,}400{,}000 \text{ lb./sq. in.}$$

and $S = 900$ lb./sq. in. E_r represents Euler's working formula for *round-ended* wooden columns with a factor of safety of $n = 3$. That is, E_r represents the equation*

$$\frac{P}{A} = \frac{\pi^2 E}{3\left(\frac{L}{k}\right)^2} = \frac{\pi^2 E}{36\left(\frac{L}{d}\right)^2} = \frac{\pi^2 \times 1{,}400{,}000}{36\left(\frac{L}{d}\right)^2}. \tag{b}$$

The curve F represents Eq. (19). This curve is tangent to Euler's curve at C where $P/A = \frac{2}{3}S$. Hence, if by the use of the F.P.L. formula [Eq. (19)] $P/A < \frac{2}{3}S$, the results should be discarded and Euler's formula [Eq. (*b*)] should be used.

Note that Euler's formula for round ends is used and that therefore, the F.P.L. formula is conservative.

As stated above, by the use of tables, the F.P.L. formula [Eq. (19)] is readily used for any specified kind and grade of wood. For general use, however, a straight line formula is preferable. Draw the straight line BC (Fig. 424). The equation of this line is (for select hemlock)

* Do not confuse K and k. K is defined by Eq. (a), while k is the radius of gyration of the section. For a rectangular section, $k^2 = d^2/12$. Hence

$$\left(\frac{L}{k}\right)^2 = 12\left(\frac{L}{d}\right)^2.$$

$$\frac{P}{A} = 1100 - 20\frac{L}{d} = 1100 - 5.8\frac{L}{k} \text{ (not a legal formula).} \quad (20)$$

$$10 < \frac{L}{d} < 25. \qquad 34.6 < \frac{L}{k} < 87.$$

$$\text{Max. } \frac{P}{A} = 900. \qquad E = 1{,}400{,}000 \text{ lb./sq. in.}$$

If $L/d < 10$, no column formula is needed. The member is designed as a short prism with 900 lb./sq. in. as the allowable unit stress. If $L/d > 25$, Euler's formula [Eq. (*b*)] is used.

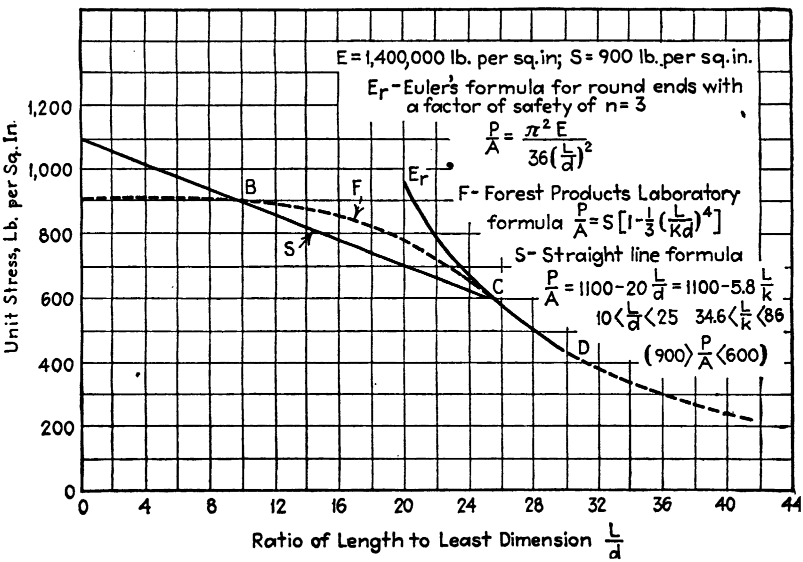


FIG. 424.

Note.—For a good grade of wood, kind not specified, formula (20) may be used.

In a similar manner, straight line formulas may be derived for the various kinds and grades of wood.

For Select Oak:

$$\frac{P}{A} = 1200 - 20\frac{L}{d} \text{ (N. Y. law).} \quad (21)$$

$$10 < \frac{L}{d} < 25.$$

$$\text{Max } \frac{P}{A} = 1000. \qquad E = 1{,}500{,}000 \text{ lb./sq. in.}$$

273. Note.—It is frequently convenient to let f_c designate the value of the right-hand member of a column formula. If this is done we may put

$$\frac{P}{A} = f_c.$$

Hence, f_c having been evaluated for a particular value of L/k (by means of a column formula), the sectional area required for the column to carry the load P is

$$A = \frac{P}{f_c}.$$

274. Résumé. Working Formulas.—For the same kind of material and the same kind of construction, various working formulas were used in the past. Almost every principal city had its own formulas, and the formulas of two cities sometimes differed widely. Since 1934 there has been a strong tendency toward standardization of formulas. In the following formulas, when two sets of upper limits are given for L/k, the first set is that determined theoretically and the second set gives the maximum limits specified in practice (see Art. 276).

1. *Structural Steel.*

a. Railway Bridges:

$$\frac{P}{A} = 15{,}000 - \frac{1}{4}\left(\frac{L}{k}\right)^2 \quad \text{Riveted ends} \qquad (22)$$

$$\frac{L}{k} \leq 140$$

$$\frac{P}{A} = 15{,}000 - \frac{1}{3}\left(\frac{L}{k}\right)^2 \quad \text{Pin ends} \qquad (23)$$

b. Buildings or any structure for which the live and the impact load are fairly definitely defined:

$$\frac{P}{A} = 17{,}000 - 0.485\left(\frac{L}{k}\right)^2 \quad \text{(A.I.S.C.)} \qquad (24)$$

$$\frac{L}{k} \leq 120$$

$$\frac{P}{A} = \frac{18{,}000}{1 + \dfrac{1}{18{,}000}\left(\dfrac{L}{k}\right)^2} \quad \text{(A.I.S.C.)} \qquad (25)$$

$$\frac{L}{k} > 60 \qquad \frac{L}{k} > 120$$

2. *Cast Iron:*

$$\frac{P}{A} = 9000 - 40\frac{L}{k} \text{ (N. Y. law).} \tag{26}$$

$$\frac{L}{k} < 125. \qquad \text{Max. } \frac{L}{k} = 90 \text{ (Note, Art. 271).}$$

3. *Wood.*

a. Good Grade, Kind Not Specified:

$$\frac{P}{A} = 1100 - 20\frac{L}{d} = 1100 - 5.8\frac{L}{k}. \tag{27}$$

$$10 < \frac{L}{d} < 25. \qquad 34.6 < \frac{L}{k} < 87.$$

$$\text{Max. } \frac{P}{A} = 900 \text{ lb./sq. in.} \qquad E = 1{,}400{,}000 \text{ lb./sq. in.}$$

b. Select Oak:

$$\frac{P}{A} = 1200 - 20\frac{L}{d} = 1200 - 5.8\frac{L}{k} \text{ (N. Y. law).} \tag{28}$$

$$10 < \frac{L}{d} < 25. \qquad 34.6 < \frac{L}{k} < 87.$$

$$\text{Max. } \frac{P}{A} = 1000. \qquad E = 1{,}500{,}000 \text{ lb./sq. in.}$$

Note.—It is necessary to be consistent in the use of units. If k (or d) is expressed in inches, L must be expressed in inches when determining L/k (or L/d). In column formulas the inch and the pound are commonly used as units.

Example I.—A 12- by 12-in. wooden column is 8 ft. long. What central load may be applied?

$$\frac{L}{d} = \frac{8 \times 12}{12} = 8 < 10. \quad \text{(No column formula is needed.)}$$

Putting $P/A = 900$,

$$P = 900 \times 144 = 128{,}600 \text{ lb.}$$

Example II.—If the column of Example I is 15 ft. long, find P.

$$\frac{L}{d} = \frac{15 \times 12}{12} = 15 > 10.$$

Hence [formula (27)],

$$\frac{P}{144} = 1100 - 20 \times 15 = 800$$

Therefore

$$P = 800 \times 144 = 115{,}200 \text{ lb.}$$

Example III.—A standard I-beam, 12 in. 31.8 lb., is to be used as a column in a building. The column is 10 ft. long. What central load may be applied if the column is not supported laterally?

$$A = 9.26, \qquad k = 1.01, \qquad \frac{L}{k} = \frac{120}{1.01} = 119 < 120.$$

$$\frac{P}{9.26} = 17{,}000 - 0.485(119)^2 = 10{,}130 \text{ lb./sq. in.}$$

Therefore

$$P = 93{,}800 \text{ lb.}$$

PROBLEMS

243. An 8-in. 23-lb. standard steel I-beam, 16 ft. long, is used as a column. This column is laterally unsupported. Show that Euler's formula is applicable (Fig. 422). Find the safe load if a factor of safety of $n = 2.5$ is used. *Ans.* 22,900 lb.

244. If the column of Problem 243 is laterally supported so that it must bend about the 1 . . . . 1 axis (perpendicular to the web), find the safe load if a factor of safety of 2.5 is used [Eq. (16)]. *Ans.* 92,000 lb.

245. If the column of Problem 243 is firmly held at the ends so that it may be considered fixed ended, find the safe load (Art. 269). *Ans.* 97,900 lb.

246. Find the load a 10- by 10-in. wooden post, 20 ft. long, can safely carry if laterally unsupported [Eq. (27)]. *Ans.* 62,000 lb.

247. If the post described in Problem 246 is laterally braced at mid height, find the safe load. *Ans.* 86,000 lb.

248. A wooden post, 8 by 10 in. (nominal size), is 10 ft. long and is unsupported laterally. Find the load [Eq. (27)]. *Ans.* 64,000 lb.

249. Find the load for the actual commercial size of the post considered in Problem 248. The actual size of a timber nominally 8 by 10 in. is 7.5 by 9.5 in., allowance being made for the width of the saw kerf and for shrinkage of the wood. By what percentage is the amount of the load for Problem 248 reduced? *Ans.* 55,575 lb.; 13 per cent.

275. Economy of Material.—1. A column free to bend in any direction will bend in the direction of the *least* radius of gyration. If the radius of gyration is the same in all directions, the column is equally strong in all directions.

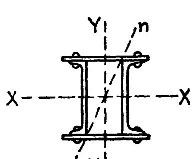


FIG. 425.

Let X and Y (Fig. 425) be two *axes of symmetry* and let I_x and I_y, respectively, be the moments of inertia of the section with respect to these axes. In Elementary Mechanics it is shown (X and Y being axes of symmetry) that, if $I_x = I_y = I$, say, then the moment of inertia of the section with respect to any gravity

axis is I. For instance, if $I_x = I_y = I$, then $I_n = I$ where n is any gravity axis of the section (see Art. 155).

Assume now that I is the same for all gravity axes. Since $I = Ak^2$, it follows that k is the same for all gravity axes and, therefore, is the same in all directions. Hence

I. If $I_x = I_y$ (X and Y being axes of symmetry), *the column is equally strong in all directions.*

It can also be shown that

II. If $I_x < I_y$ (X and Y being axes of symmetry), *then* k_x *is the least radius of gyration of the section.*

In designing a column, it is frequently advantageous (as far as economy of material is concerned) to make $I_x = I_y$. That is, for a given value of the load P it is frequently possible to reduce the required sectional area A by making $I_x = I_y$.

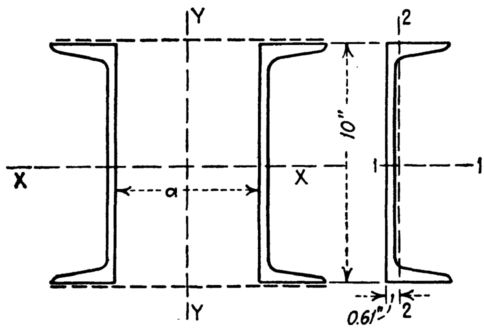


FIG. 426.

Example.—A column is to consist of two 10-in. 20-lb. channels latticed together as shown in Fig. 426. Find the distance between channels, back to back, so that $I_x = I_y$.

For each channel $I_1 = 78.5$, $I_2 = 2.80$, $A = 5.86$. Distance of center of gravity of channel from back $= 0.61$ in. (see figure). Therefore

$$I_x = 2 \times 78.5.$$

To find I_y use the theorem that

$$I_y = \Sigma(I_g + Ad^2)\text{(Art. 137).}$$

$$I_y = 2\left[I_2 + A\left(\frac{a}{2} + 0.61\right)^2\right] = 2\left[2.80 + 5.86\left(\frac{a}{2} + 0.61\right)^2\right].$$

Equating I_x to I_y and solving for a,

$$a = 6 \text{ in.}$$

2. For a given sectional area A, the strength of the column is increased if k is increased. Now k may be increased by placing the material as far from the axis of the column as is practicable. For instance, a cast-iron column is hollow. Such a column is stronger (and stiffer) than a solid column of the same sectional area.

276. Practical Limits for L/k.—If L/k is large (if k is small relative to L), the column may lack the rigidity deemed necessary. A slender column is liable to bend if subjected to an accidental lateral pressure. Moreover, in the case of impact, such a column may be subject to excessive vibrations. Hence an upper limit to L/k often is specified. For instance, the A.I.S.C. specifies that for main members of a building max. $L/k = (120)$ [see Eq. (24)]. Note that *theoretically* the A.I.S.C. formula [Eq. (25)] holds for $L/k > 60$.

On the other hand, if L/k is small (if k is large relatively,) the material may be so placed that it is too thin and hence liable to buckle, particularly if subjected to an accidental lateral pressure. Moreover, practical considerations often limit the size of column.

All things considered, *a column of moderate slenderness ratio may be the most advantageous.*

277. Design of a Column.—If a column is to be designed, it will be necessary to determine the area and the shape of a section so that A and k will satisfy a specified column formula. If the section is to be round or square, the value of A and of k may be expressed in terms of some dimension d of the section. Thus, if the section is to be square, $A = d^2$ and $k = d/\sqrt{12}$ where d is the side of the square (Art. 259). By substituting these expressions for A and k in the column formula, the resulting equation may be solved for d.

When a column is to consist of *one* or *more* structural shapes, it is not possible as a rule to find expressions for A and k in terms of some one dimension of the shape or shapes. In such cases the method of trial may be used. That is, assume a section and substitute the values of A and of k in a column formula. If the value of P thus determined is larger or smaller than that required for the column to carry, make another trial. Repeat until a suitable section is found.

Note 1.—In practice, the column formula usually is specified.

Note 2.—If only commercial sizes of the material are available, select the nearest commercial size above that theoretically required.

Note 3.—If a column is free to bend in all directions, k is the least radius of gyration of the section.

An experienced designer develops methods of his own for the design of columns. For the beginner, the method of procedure used in Examples I and II will probably be found the most convenient. Note that in these examples *the minimum sectional area of the member is first determined on the assumption that the member is a short prism.*

Example I.—A wooden column, 11 ft. long, is to carry a load of 30,000 lb. Design the column (square). Use formula (27) of Art. 274.

For a short prism, $P/A = 900$.

Or

$$A = \frac{30{,}000}{900} = 33.3.$$

Nearest commercial size above is a 6 by 6 in.

Try a 6- by 6-in. section. $A = 36$, $d = 6$.

$$\frac{L}{d} = \frac{11 \times 12}{6} = 22.$$

Since $L/d > 10$, the column formula must be used.

Therefore

$$\frac{P}{36} = 1100 - 20 \times 22 = 660.$$

Or

$$P = 23{,}750 < 30{,}000.$$

The section is too small.

Try an 8 by 8 in. $A = 64$, $d = 8$.

$$\frac{L}{d} = \frac{11 \times 12}{8} = 16.5 > 10.$$

Therefore

$$\frac{P}{64} = 1100 - 20 \times 16.5 = 770.$$

Or

$$P = 49{,}200 > 30{,}000. \quad \text{(Too large.)}$$

Take an 8 by 8 in. since there is no commercial section between a 6 by 6 in. and an 8 by 8 in.

Note.—Since the column is to be square, the size of the column may be directly determined. If d is the lateral dimension of the column, Eq. (27) may be written

$$\frac{30{,}000}{d^2} = 1100 - 20 \times \frac{11 \times 12}{d}.$$

Simplifying,

$$d^2 - 2.4d - 27.5 = 0 \qquad \text{or} \qquad d = 6.50 \text{ in.}$$

since $L/d = 11 \times 12/6.50 = 20.6 > 10$, the column formula is applicable.

Example II.—A standard I-beam is to be used as a column in a building [formula (24)]. The column is to be 25 ft. long and laterally supported in such a way that it will bend about the 1 . . . 1 axis. Design the column to carry a load of 120,000 lb.

For a short prism, $P/A = 15{,}000$. Or

$$A = \frac{120{,}000}{15{,}000} = 8.$$

The nearest commercial size above (of least weight) is a 10-in. 30-lb. beam. (Consult Table III for properties of this section.)

Try a 10-in. 30-lb. beam. $A = 8.75$, $k = 3.91$.

$$\frac{L}{k} = \frac{25 \times 12}{3.91} = 76.8.$$

Therefore

$$\frac{P}{8.75} = 17{,}000 - 0.485(76.8)^2 = 14{,}140$$

Or

$$P = 123{,}700 > 120{,}000.$$

Note.—Handbook tables give the value of

$$f_c = 17{,}000 - 0.485\left(\frac{L}{k}\right)^2 \qquad \text{or} \qquad f_c = \frac{18{,}000}{1 + \frac{1}{18{,}000}\left(\frac{L}{k}\right)^2}$$

for values of L/k between 1 and 200. The design of a column (or beam, etc.) is often simplified if use is made of tables.

If a laterally unsupported I-beam, channel, or angle is to be used as a column, the L/k is apt to be large since minimum k

for any one of these sections, as a rule, is relatively small. In such cases, the method of design often may be simplified by *assuming, as a first trial, a section for which L/k equals the maximum value allowed.*

Example III.—An angle with equal legs is to be used as a column in a bridge [formula (22), Art. 274]. The column is to be 10 ft. long and is to be used as a secondary member (max. $L/k = 120$). Design the column to carry a load of 45,000 lb.

Put $L/k = 120$ and solve for k. That is, $10 \times 12/k = 120$. Or $k = 1$. As a first trial select an angle of minimum weight whose $k \gtreqless 1$. Try a 6- by 6- by $\frac{3}{8}$-in. angle. $A = 4.36, k = 1.19$ (axis 3 . . . 3).

$$\frac{L}{k} = \frac{120}{1.19} = 100.8$$

Therefore

$$\frac{P}{4.36} = 15{,}000 - \frac{1}{4}(100.8)^2 = 12{,}460.$$

Or

$$P = 54{,}300 \text{ lb.} > 45{,}000. \quad \text{(Too large.)}$$

Try a 5- by 5- by $\frac{7}{16}$-in. angle. $A = 4.18, k = 0.98,$

$$L/k = 122.4.$$

$$\frac{P}{5.06} = 15{,}000 - \frac{1}{4}(122.4)^2 = 11{,}250$$

Or

$$P = 47{,}000 \text{ lb.} > 45{,}000.$$

Select a 5- by 5- by $\frac{7}{16}$-in. angle.

PROBLEMS

250. A square wooden column, 15 ft. long, is to be used to carry a load of $P = 70{,}000$ lb. Design the column [Eq. (27)]. *Ans.* 10 in. square.

251. A standard steel I-beam, 23 ft. long, is to be used as a column in a building. The column is to be laterally supported and is to carry a load of $P = 150{,}000$ lb. Design the column. *Ans.* 12-in. 35-lb. beam.

252. A structural-steel angle, 8 ft. 4 in. long, is to be used as a column in a building; the column is unsupported laterally. The column is a secondary member and is to be designed for a load of $P = 15{,}000$ lb.

Ans. 4 by 4 by $\frac{1}{4}$ in.

278. The *compression flange* of a beam acts as a column. If then the unbraced length of a beam is excessive, the flange may buckle (Fig. 427). Hence for a given fiber stress a limitation

must be placed upon the unbraced length of a beam. In practice, as a rule, long beams are laterally braced.

The *vertical strip* AB of the web under a load or over a reaction (Fig. 428) acts as a column. If the unit compressive stress in this strip is excessive, the strip may buckle. It may be necessary therefore to reinforce the web under a concentrated load or over a reaction.

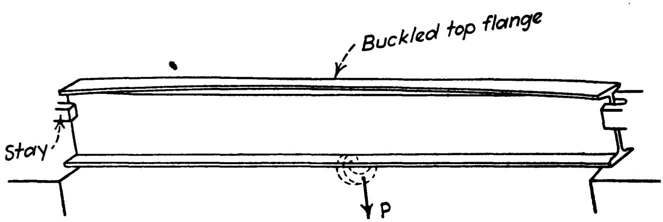


FIG. 427.

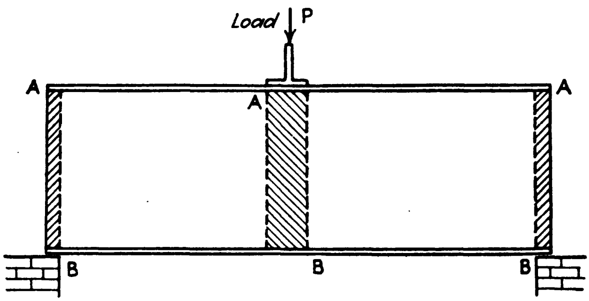


FIG. 428.

The student should consult his handbook ("A.I.S.C. Manual," for instance) for details regarding unsupported compression flanges and stiffener angles.

COLUMN AND BEAM ACTION COMBINED

279. Method of Adding Areas.—A member subjected to a transverse load Q and an axial load P (Fig. 429) may be called *a beam subjected to column action* or a *column subjected to beam action.*

The theoretically correct treatment of a member subjected to both beam and column action leads to a formula so cumbersome and involved that it is of little practical use. In practice an approximate formula is used. The *method of adding areas* will now be explained.

It will be assumed that the member tends to bend in the direction of the transverse load Q. Hence k equals the radius of gyration of the member in the direction of Q.

Given two members, both of length L (Fig. 430). Assume both members rectangular and of height h so that for both members $k = h/\sqrt{12}$ (Art. 259) and $c = h/2$. Note that k and c are functions of h (and not of the width of the members). Member a is to carry a transverse load Q, and member b an axial load P.

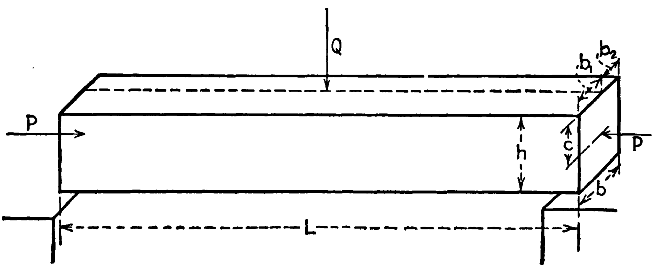


FIG. 429.

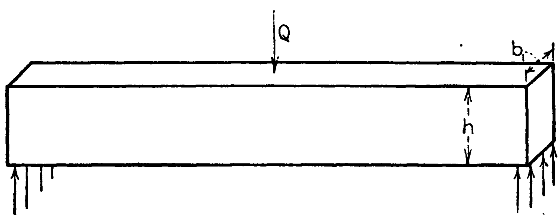


FIG. 430*a*.

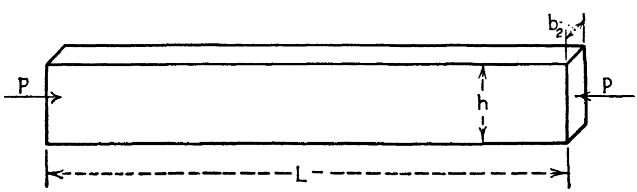


FIG. 430*b*.

Consider the beam (Fig. 430*a*). Let S equal the safe unit stress in the outer fiber. Let M equal the maximum bending moment due to the transverse load Q. Since $M = SI/c = SAk^2/c$, the area required to carry safely the load Q is (solving for A and designating it by A_b)

$$A_b = \frac{Mc}{Sk^2}. \tag{a}$$

That is, for given values of k and c, Eq. (a) determines the sectional area required for the beam to carry the load Q.

Consider now the column (Fig. 430b). If f_c is the safe average unit stress for a column of the given slenderness ratio L/k, then the area required for the column to carry the load P is

$$A_c = \frac{P}{f_c}. \tag{b}$$

To determine f_c, a column formula must be used.

If the two members are placed side by side and fastened together (glued together, say, in case of rectangular sections, Fig. 429) so that they will form a single member whose sectional area is

$$A = A_b + A_c = \frac{Mc}{Sk^2} + \frac{P}{f_c}, \tag{29}$$

this member ought to be able to carry both the transverse load Q and the axial load P.

Assume now that it is required to design a member that is to sustain a transverse load Q and an axial load P. The foregoing discussion suggests the following method of procedure:

1. Select a trial member. Let k equal the radius of gyration of this member in the direction of Q. If the section is to be rectangular, it is more convenient to assume the height h of the member.

2. Find A_b, the area required to resist the beam action.

$$A_b = \frac{Mc}{Sk^2} \tag{c}$$

where M = maximum bending moment in the member due to the transverse load Q.
S = allowable unit stress in the outer fibers of a beam of the given material.
c = distance of outermost fiber from the neutral axis.

3. Find f_c, the allowable average unit stress for a column of the given slenderness ratio L/k (as determined by a suitable column formula). Then find A_c, the area required to resist column action,

$$A_c = \frac{P}{f_c}. \tag{d}$$

4. For a safe and economical member, A, the area of the member, should not be less than $A_b + A_c$ and should be as nearly equal to $A_b + A_c$ as standard sizes of sections will permit. As a rule, several trials must be made before a satisfactory member is obtained.

Note.—If the beam action (or the column action) predominates, a first approximation to the required size of the member may be obtained by considering the beam action (or the column action) alone.

Example I.—A wooden member, 14 ft. long, is to carry a uniformly distributed load of $W = 9600$ lb. and an axial load of $P = 2{,}400$ lb. (Fig. 431). Take $S = 1200$ lb./sq. in. Let

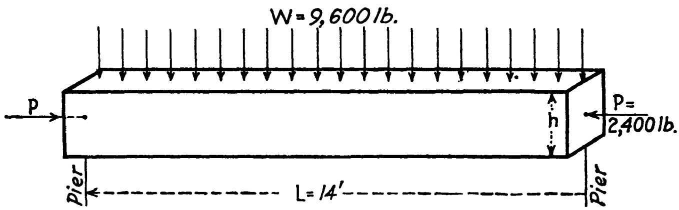


Fig. 431.

$b =$ about $\frac{2}{3}h$. Design the member. An experienced designer would know that in this case beam action strongly predominates. Hence a first approximation may be obtained by considering beam action alone.

Beam Action:

$$M = \tfrac{1}{8}WL = \tfrac{1}{8} \times 9600 \times 14 \times 12 = 201{,}600 \text{ in.-lb.}$$

Since $M = SI/c$,

$$201{,}600 = 1200\frac{I}{c}.$$

Therefore

$$\frac{I}{c} = 168.$$

For a rectangular section (putting $b = \frac{2}{3}h$),

$$\frac{I}{c} = \frac{1}{6}bh^2 = \frac{1}{9}h^3 = 168 \qquad \text{or} \qquad h = 11.5 \text{ in.}$$

Try $h = 12$ in. Therefore

$$k = \frac{12}{\sqrt{12}} = 3.46, \qquad k^2 = 12, \qquad \text{and} \qquad \frac{L}{k} = \frac{14 \times 12}{3.46} = 48.6.$$

Area required to resist beam action is [Eq. (*c*)]

$$A_b = \frac{Mc}{Sk^2} = \frac{201{,}600 \times 6}{1200 \times 12} = 84.$$

Column Action.—Using formula 27 (Art. 274),

$$f_c = 1100 - 5.8 \times 48.6 = 818.$$

Area required to resist column action is [Eq. (*d*)]

$$A_c = \frac{P}{f_c} = \frac{2400}{818} = 2.94.$$

Hence area required to resist both beam action and column action is

$$A = A_b + A_c = 84.00 + 2.94 = 86.94 \text{ sq. in.}$$

Or, since $A = bh = 12b$ (for the trial member, $h = 12$),

$$b = \frac{86.94}{12} = 7.24 \text{ in.}$$

Nearest commercial size is 8 by 12 in.

Example II.—A steel member is to be subjected to a maximum bending moment of 1,440,000 in.-lb. and an axial thrust of 60,000 lb. The member is to be 20 ft. long and is to consist of two channels with flanges turned in (Fig. 432). The transverse

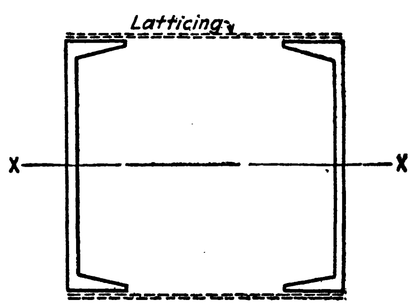


FIG. 432.

load producing the bending moment is to act at right angles to the axis $X \ldots X$. It is required to select the channels. Take $S = 18{,}000$ lb./sq. in. The A.I.S.C. column formula is to be used [Eq. (24), Art. 274].

For a first approximation to the size of the channel, consider beam action alone.

$$M = \frac{SI}{c} \qquad \text{or} \qquad 1{,}440{,}000 = 18{,}000\frac{I}{c}.$$

Therefore

$$\frac{I}{c} = 80.$$

Since there are two channels, the I/c for each channel is 40. Try a 15-in. 45-lb. channel. For each channel,

$$A = 13.17, \qquad \frac{I}{c} = 49.8, \qquad k = 5.33, \qquad k^2 = 28.4,$$

$$\frac{L}{k} = \frac{20 \times 12}{5.33} = 45.$$

For the two channels,*

$$A = 26.34, \qquad \frac{I}{c} = 99.6, \qquad k = 5.33, \qquad k^2 = 28.4, \qquad \frac{L}{k} = 45.$$

Therefore, the total area required for beam action is

$$A_b = \frac{Mc}{Sk^2} = \frac{1{,}440{,}000 \times 7.5}{18{,}000 \times 28.4} = 21.10 \text{ sq. in.}$$

Next consider column action.

$$f_c = 17{,}000 - 0.485(45)^2 = 16{,}000$$

$$A_c = \frac{60{,}000}{16{,}000} = 3.75 \text{ sq. in.}$$

Now $A > A_b + A_c$. That is, $26.34 > 21.1 + 3.75$. The channels selected are satisfactory.

In the solution of this problem, the channels were actually decided upon on the third trial.

Problem 253.—A standard I-beam, 20 ft. long, is to carry a uniformly distributed load of $W = 32{,}000$ lb. and an axial load of $P = 32{,}000$ lb. Take $S = 18{,}000$ lb./sq. in. The A.I.S.C. column formula is to be used [Eq. (24), Art. 274]. Design the member. *Ans.* 15-in. 50-lb. I-beam.

280. Column Subjected to an Axial Load and an Eccentric Load.†—Let a column be subjected to an axial load P and an eccentric load Q (Fig. 433). Let Q be applied to the side of the column by means of a bracket as shown, and assume that Q

* For one channel, $k^2 = I/A$. For the two channels, $k^2 = 2I/2A = I/A$. Hence the k for the two channels (with respect to any axis of symmetry) is the same as that for one channel.

† Assume eccentric load to act in a principal plane (a plane of symmetry always is a principal plane, Art. 156).

acts in a plane of symmetry. Since commercial column formulas assume that the column is pin ended, let P be the vertical pressure of the pin on the column.

Replace the load Q by an axial load Q' (equal to Q) and a couple QQ whose moment is $M = Qe$. There are then three actions to be considered:

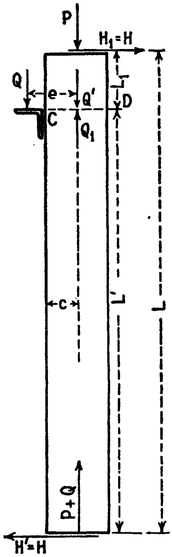


FIG. 433.

1. Column action due to P.
2. Column action due to Q' ($Q' = Q$).
3. Beam action due to the couple QQ_1 whose moment is $M = Qe$.

Let L = length of column between pins.

L' = length below the section CD.

L_1 = length above the section CD.

λ = greater length, L' or L_1.

f_c = safe average unit stress (as determined by a column formula) for the column of length L.

f'_c = safe average unit stress for the column of length L'.

Figure 433 represents the column as a free body. By putting $\Sigma F_x = 0$, $H_1 = H' = 0$. Therefore $H_1 = H' = H$ (say). Putting Σ moments about the lower pin $= 0$,

$$HL - Qe = 0.$$

Therefore

$$H = \frac{Qe}{L}. \tag{a}$$

Hence the internal moment just above section CD is

$$M_1 = H_1L_1 = HL_1 = \frac{L_1}{L}Qe.$$

The internal moment just below section CD is

$$M' = H'L' = HL' = \frac{L'}{L}Qe.$$

The maximum moment in the column is, therefore,

$$M_m = \frac{\lambda}{L}Qe,$$

where λ = greater of the two lengths, L_1 or L'.

The column must have a sectional area A sufficient to withstand the three actions. The area required to withstand the column action due to P is

$$A_c = \frac{P}{f_c}.$$

Area required to withstand the column action due to Q' is ($Q' = Q$)

$$A'_c = \frac{Q}{f'_c}.$$

Area required to withstand the beam action due to Q is

$$A_b = \frac{M_m c}{Sk^2} = \frac{\lambda}{L}\frac{Qec}{Sk^2}.$$

Hence the total area required is

$$A = A_c + A'_c + A_b = \frac{P}{f_c} + \frac{Q}{f'_c} + \frac{\lambda}{L}\frac{Qec}{Sk^2}. \tag{30}$$

Note.—1. If the eccentric load acts near the upper end, we may put $L' = L = \lambda$ and $f'_c = f_c$. Therefore

$$A = \frac{P + Q}{f_c} + \frac{Qec}{Sk^2}. \tag{31}$$

2. If the eccentric load acts near the middle of the column, put $\lambda = L/2$. Therefore

$$A = \frac{P}{f_c} + \frac{Q}{f'_c} + \frac{Qec}{2Sk^2}. \tag{32}$$

Example. Fig. 433.—A wooden column 12 by 12 in. is 16 ft. long. It carries an axial end load of $P = 60{,}000$ lb. An eccentric load Q is to be applied 4 ft. from the upper end with an eccentricity of $e = 8$ in. Taking $S = 900$ lb./sq. in., find Q.

$L = 192$ in., $L' = 144$ in., $k = \sqrt{12} = 3.46$, $A = 144$ sq. in., $L/k = 192/3.46 = 55.5$, $L'/k = 144/3.46 = 41.6$, $\lambda/L = \frac{12}{16} = \frac{3}{4}$. Using formula (27), Art. 274,

$$f_c = 1100 - 5.8 \times 55.5 = 778,$$
$$f'_c = 1100 - 5.8 \times 41.6 = 858.$$

There [Eq. (30)],

$$144 = \frac{60{,}000}{778} + \frac{Q}{858} + \frac{3}{4}\,\frac{Q \times 8 \times 6}{900 \times 12}.$$

Or

$$Q = 14{,}820 \text{ lb.}$$

281. Secant Formula.—A theoretically correct expression for the maximum unit stress induced in an eccentrically loaded column will now be derived for the particular case shown in Fig. 434. The forces Q and Q are applied at a distance e from the centroids of the end sections and lie in a plane of symmetry (or a principal plane).

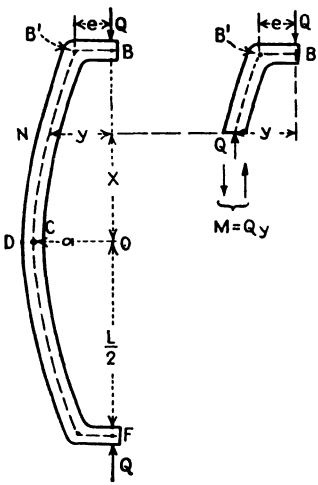


Fig. 434.

The maximum unit stress occurs in the outer fibers at C and is (Art. 261)

$$S = \frac{Q}{A} + \frac{Mc}{I} = \frac{Q}{A} + \frac{Qac}{Ak^2}$$
$$= \frac{Q}{A}\left(1 + \frac{ac}{k^2}\right) \qquad (a)$$

where a = distance of the line of action of Q from the centroid of the section CD.

Take the point O, the middle of the line FB, as the origin. Take y as positive to the left and x as positive upward. Consider the point N. The bending moment at N is (see Art. 255)

$$M = -Qy.$$

Hence

$$EI\frac{d^2y}{dx^2} = -Qy.$$

This equation may be written

$$EI\frac{d}{dx}\left(\frac{dy}{dx}\right) = -Qy$$

or

$$EId\left(\frac{dy}{dx}\right) = -Qydx.$$

Multiply through by dy/dx,

$$EI\frac{dy}{dx}d\left(\frac{dy}{dx}\right) = -Qydy.$$

Integrating,

$$\frac{EI}{2}\left(\frac{dy}{dx}\right)^2 = \frac{-Qy^2}{2} + C_1.$$

At D, $dy/dx = 0$; $y = a$. Substituting,

$$0 = \frac{-Qa^2}{2} + C_1 \qquad \text{or} \qquad C_1 = \frac{Qa^2}{2}.$$

Therefore

$$EI\left(\frac{dy}{dx}\right)^2 = Q(a^2 - y^2).$$

Solving for dy/dx,

$$\frac{dy}{dx} = \sqrt{\frac{Q}{EI}}\sqrt{a^2 - y^2}.$$

Separating variables,

$$\frac{dy}{\sqrt{a^2 - y^2}} = \sqrt{\frac{Q}{EI}}\,dx.$$

Integrating,

$$\sin^{-1}\frac{y}{a} = x\sqrt{\frac{Q}{EI}} + C_2. \tag{b}$$

Note that Eq. (*b*) is the same as Eq. (*d*) of Art. 255. To determine the constant C_2, substitute simultaneous values of x and y. At O, $x = 0$ and $y = a$.
Therefore

$$\sin^{-1} 1 = C_2 \qquad \text{or} \qquad C_2 = \frac{\pi}{2}.$$

Substituting in Eq. (*b*),

$$\sin^{-1}\frac{y}{a} = x\sqrt{\frac{Q}{EI}} + \frac{\pi}{2}.$$

Or

$$y = a\sin\left(x\sqrt{\frac{Q}{EI}} + \frac{\pi}{2}\right) = a\cos\left(x\sqrt{\frac{Q}{EI}}\right). \tag{c}$$

At B, $y = e$ and $x = L/2$. Substituting in Eq. (*c*) and solving

for a,

$$a = \frac{e}{\cos\left(\frac{L}{2}\sqrt{\frac{Q}{EI}}\right)} = e \sec\left(\frac{L}{2}\sqrt{\frac{Q}{EI}}\right).* \qquad (d)$$

Substituting this value of a in Eq. (a),

$$S = \frac{Q}{A}\left[1 + \frac{ec}{k^2} \sec\left(\frac{L}{2}\sqrt{\frac{Q}{EI}}\right)\right].$$

Or, as it is commonly written (since $I = Ak^2$),

$$S = \frac{Q}{A}\left[1 + \frac{ec}{k^2} \sec\left(\frac{L}{2k}\sqrt{\frac{Q}{EA}}\right)\right]. \qquad (33)$$

Equation (33) is known as the *secant formula*. Within the elastic limit this formula is theoretically correct for the eccentrically loaded column shown in Fig. 434.

Example.—A 12- by 12-in. wooden column, 16 ft. long, is loaded as shown in Fig. 434. $e = 8$ in., $S = 900$ lb./sq. in., $E = 1{,}400{,}000$ lb./sq. in.

1. Determine Q by the method of adding areas.
2. With the value of Q thus determined, find maximum unit stress induced in the column by means of the secant formula.

(1) $L = 192$ in., $k = \sqrt{12} = 3.46$, $k^2 = 12$,

$$\frac{L}{k} = \frac{192}{3.46} = 55.5.$$

By means of formula (27) of Art. 274,

$$f_c = 1100 - 5.8 \times 55.5 = 778.$$

Hence [Eq. (31), Art. 280, putting $P = 0$],

$$144 = \frac{Q}{778} + \frac{Q \times 8 \times 6}{900 \times 12}.$$

Or

$$Q = 25{,}150 \text{ lb.}$$

(2) $$\frac{L}{2k}\sqrt{\frac{Q}{EA}} = \frac{55.5}{2}\sqrt{\frac{25{,}150}{1{,}400{,}000 \times 144}} = 0.308 \text{ rad.} = 17.64°.$$

* In Art. 255 it was seen that, for a centrally loaded column, the critical load is obtained when $\frac{L}{2}\sqrt{\frac{Q}{EI}} = \frac{\pi}{2}$. If in Eq. ($d$) we put $e = 0$ and $\frac{L}{2}\sqrt{\frac{Q}{EI}} = \frac{\pi}{2}$, $a = \frac{0}{0}$. That is, for a centrally loaded column, a is indeterminate.

sec $17.64° = 1.048$. Therefore [Eq. (33)],

$$S = \frac{25{,}150}{144}\left[1 + \frac{8 \times 6}{12} \times 1.048\right] = 906 \text{ lb./sq. in.}$$

SHORT PRISM SUBJECTED TO AXIAL AND BENDING STRESSES

282. In Art. 274, column formulas for supposedly centrally loaded columns are given. Note that for each column formula there is a value for L/k (or L/d), such that if L/k (or L/d) is less than this value the member may be treated as a short prism or block with a given allowable maximum compressive stress.

In practice it frequently occurs that a compressive member may be treated as a short prism. In the remaining articles of this chapter, a short prism subjected to both axial and flexural stresses will be given further consideration.

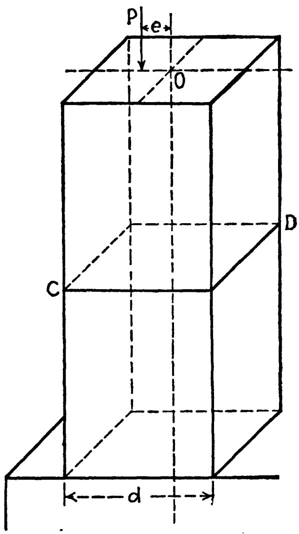


FIG. 435.

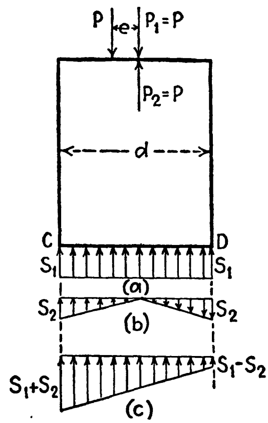


FIG. 436.

283. Eccentric Load in Plane of Symmetry. Fig. 435.—Replace the load P by an axial load P_1 equal to P, and a couple whose moment is $M = Pe$ (Fig. 436). Note that the deflection of a short prism may be neglected. Hence the stress on a section CD may be considered as composed of a direct stress whose intensity is (Fig. 436a)

$$S_1 = \frac{P}{A}$$

and a bending stress whose maximum intensity is (Fig. 436*b*)

$$S_2 = \frac{Mc}{I} = \frac{Pec}{I}.$$

Combining the two stresses (Fig. 436*c*), we obtain the resultant stress acting on the section *CD* whose maximum intensity is at *C*,

$$S_{\text{max.}} = S_1 + S_2,$$

and whose minimum intensity is at *D*,

$$S_{\text{min.}} = S_1 - S_2.$$

Hence the stress in the outer fiber is

$$S = S_1 \pm S_2 = \frac{P}{A} \pm \frac{Pec}{I} = \frac{P}{A}\left(1 \pm \frac{ec}{k^2}\right). \tag{34}$$

Referring to Fig. 436 [or Eq. (34)], note that if S_2 is less than S_1 the stress at *D* is compression, while if S_2 is greater than S_1 the stress is tensile. If $S_2 = S_1$, the stress at *D* is zero.

From Eq. (34), it is seen that $S_D = 0$ if $ec/k^2 = 1$, *i.e.*, if $e = k^2/c$. If, then, $e \lesseqgtr k^2/c$, there will be no tensile stress induced in the prism. This is a matter of practical importance. Brick and stone masonry and plain concrete structures have little tensile strength. Hence, to insure safety, no tensile stress should be permitted in a wall, dam, or support made of brick, stone, concrete, or any like material.

Note.—If $ec/k^2 = 1$, then by Eq. (34)

$$S_{\text{max.}} = \frac{2P}{A} \quad \text{and} \quad S_{\text{min.}} = 0. \tag{35}$$

For a *rectangular section,* $k^2 = d^2/12$ and $c = d/2$. Hence $k^2/c = d/6$. If then the line of action of *P* falls within the middle third of a rectangular section, there will be no tensile stresses induced on the section (Fig. 437).

For a *circular section,* $k^2 = r^2/4$ and $c = r$. Hence

$$\frac{k^2}{c} = \frac{r}{4} = \frac{d}{8}.$$

If then the line of action of *P* falls within the middle fourth of a circular section (if $e \lesseqgtr d/8$), there will be no *tensile* stresses induced on the section (Fig. 438).

Note.—If the eccentric load is tensile, the deflection of the member tends to lessen the effect of eccentricity (Fig. 439). The maximum stress occurs at the ends where the moment is a maximum and is equal to Pe the same as that for a short prism. Hence any *tensile* member eccentrically loaded may be treated as a short prism and Eq. (34) may be applied, due regard being given to sign.

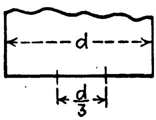


FIG. 437.

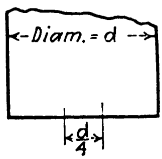


FIG. 438.

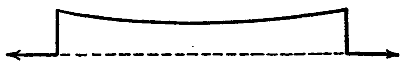


FIG. 439.

Example. Fig. 440.—A 2- by 6-in. eyebar is subjected to a pull of $P = 24$ tons whose line of action is $e = 0.8$ in. from the axis of the bar. Find the stresses in the outer fibers neglecting the deflection of the bar.

$$S_1 = \frac{P}{A} = \frac{24}{12} = 2 \text{ tons/sq. in. (tensile).}$$

$$S_2 = \frac{Pec}{I} = \frac{24 \times 0.8 \times 3}{\frac{1}{12} \times 2 \times 6^3} = 1.6 \text{ tons/sq. in.}$$

$$S = S_1 \pm S_2.$$

Therefore $S_C = 3.6$ tons/sq. in. (tensile), and $S_D = 0.4$ ton/sq. in. (tensile).

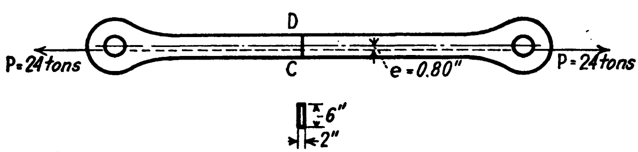


FIG. 440.

Problem 254.—A concrete foundation 4 ft. square is to sustain an eccentric load. What is the maximum eccentricity if no tensile stress is to be induced in the foundation? With this eccentricity, find the maximum load this foundation can sustain if the allowable stress is 600 lb./sq. in. [Eq. (35)].

Ans. $e = 8$ in.; $P = 691{,}000$ lb.

284. Transverse Load in Plane of Symmetry, and Axial Load. Fig. 441.—The normal stress on the section CD is a combination of the direct stress due to P and the flexural stress due to Q. If e equals the distance of point of application of Q from the

section CD, then $M = Qe'$. Note that by assumption the line of action of P goes through O (the centroid of CD) and that therefore the moment of P with respect to any axis through O is zero. Hence the maximum and minimum normal stresses on CD are

$$S = S_1 \pm S_2 = \frac{P}{A} \pm \frac{Mc}{I}$$

$$= \frac{P}{A} \pm \frac{Qe'\,c}{I}. \qquad (36)$$

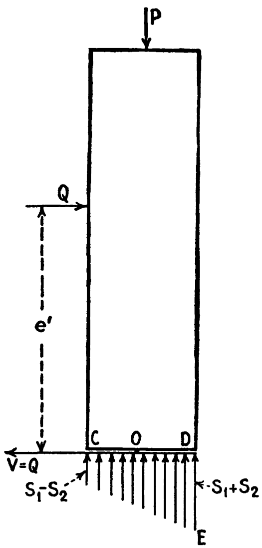


FIG. 441.

Note.—There is also a shearing stress on the section CD. If V equals the total shear on CD, $V = Q$.

Example.—A 12- by 12-in. wooden post is 6 ft. long. It carries an axial load of $P = 72{,}000$ lb. If the allowable unit normal stress is $S = 900$ lb./sq. in., what transverse load Q may be applied 1 ft. from the top?

$$S_1 = \frac{P}{A} = \frac{72{,}000}{144} = 500 \text{ lb./sq. in. (compression).}$$

$$S_2 = \frac{Qe'\,c}{I} = \frac{Q \times (5 \times 12) \times 6}{\frac{1}{12} \times 12 \times 12^3} = \frac{5}{24}Q.$$

Therefore [Eq. (36)],

$$900 = 500 + \tfrac{5}{24}Q = S_{\text{max.}}$$

or

$$Q = 1920 \text{ lb. (compression)}$$

With $Q = 1920$ lb.,

$$S_{\text{min.}} = 500 - \tfrac{5}{24} \times 1920 = 100 \text{ lb./sq. in. (compression)}$$

285. Transverse Load and Eccentric Load, Both in a Plane of Symmetry. Fig. 442.—By assumption, the deflection of the prism may be neglected. Note that the line of action of P does not go through O, the centroidal axis of CD. Hence the moment of P with respect to O is not zero. If e equals the distance of line of action of P from O (eccentricity of P), and e' equals the

distance of line of action of Q from the section CD, then the sum of the moments (with respect to O) of P and Q is (assuming P and Q to act as shown in Fig. 442)

$$M_O = Pe + Qe'.$$

Therefore

$$S = \frac{P}{A} \pm \frac{Mc}{I} = \frac{P}{A} \pm \frac{(Pe + Qe')c}{I}. \tag{37a}$$

If the direction of Q in Fig. 442 is reversed,

$$S = \frac{P}{A} \pm \frac{(Pe - Qe')c}{I}. \tag{37b}$$

PROBLEMS

255. Let Fig. 442 represent a 12- by 12-in. wooden post, 6 ft. long. The post carries an eccentric load of $P = 72{,}000$ lb. with an eccentricity of 1 in. to the left ($e = 1$ in.). A transverse load of $Q = 600$ lb. acts at the upper end also to the left ($e' = 72$ in.). Find S_C and S_D.

Ans. $S_C = 900$ lb./sq. in. (compression); $S_D = 100$ lb./sq. in. (compression).

256. In Problem 255, the direction of Q is reversed. Find S_C and S_D. *Ans.* 600 lb./sq. in.; 400 lb./sq. in.

257. In Problem 256, assume Q not given in magnitude. (All other data as given in Problem 255.) For what value of Q will the stress on CD be uniformly distributed?

Ans. $Q = 1000$ lb.; $S = P/A = 500$ lb./sq. in.

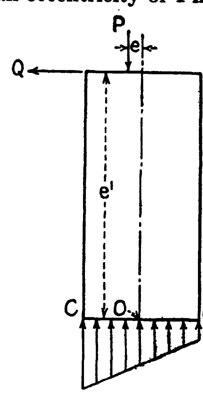


FIG. 442.

286. In a concrete or masonry dam (not reinforced) there must be no tensile stress on any section of the dam. Figure 443 represents the end view of a portion of the dam. In analyzing a dam it frequently is convenient to deal with a length of 1 ft. (1 ft. ⊥ plane of the paper). Consider now the horizontal section CD and assume that $S_C = 0$. This is the limiting case. For this case the normal stress on CD is triangularly distributed (Fig. 443). Assuming CD rectangular, the stress solid is a rectangular wedge (Fig. 444). Hence the resultant normal stress on CD goes through N, the outer edge of the middle third of CD (see Art. 10, Theorem III). Since equilibrium exists, the resultant of P (the weight of the dam) and Q (the water pressure) must also go through N (see Fig. 443). If, then, the resultant of P and Q falls within the middle third of the section CD, there will be no tensile stress on this section.

In designing a concrete dam (not reinforced) at least two conditions must be satisfied.

1. The resultant of P and Q must fall within the middle third of every section of the dam.

2. The intensity of the normal stress at D (the toe) must not exceed its allowable value.

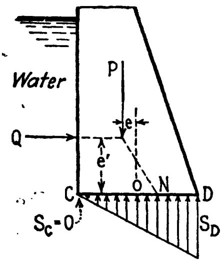


FIG. 443.

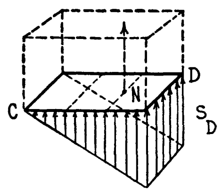


FIG. 444.

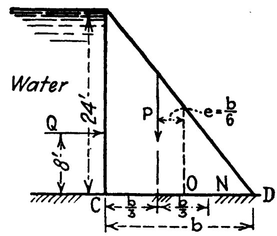


FIG. 445.

Note.—If the stress on the section is triangularly distributed, the maximum intensity of stress on the section is twice the average stress on this section. That is,

$$S_{\text{max.}} = 2S_{av.} = \frac{2P}{A} \qquad (38)$$

where P = sum of the vertical forces acting on the dam (or prism or block).

A = area of the section CD.

Example I. Fig. 445.—A triangular dam, 24 ft. high, is made of concrete (not reinforced). Water flush with top. For what width b of the section CD will the resultant of P and Q go through N, the outer edge of the middle third?

Concrete weighs 150 lb./cu. ft.

Water weighs 62.5 lb./cu. ft.

Water pressure at $C = 62.5 \times 24 = 1500$ lb./sq. ft.

$$Q = \tfrac{1500}{2} \times 24 \times 1 = 18{,}000 \text{ lb.}^*$$

$$P = 150 \times \frac{24 \times b \times 1}{2} = 1800b \text{ lb.}$$

If the resultant of P and Q goes through N, then the sum of the

* Water pressure is triangularly distributed. Hence Q equals the area of wetted surface (per 1-ft. length of dam) times the average intensity of the water pressure (Theorem IV, Art. 10).

moments of P and Q with respect to N equals zero. Therefore

$$Q \times 8 - P \times \frac{b}{3} = 0.$$

Or

$$18{,}000 \times 8 = \frac{1800b^2}{3}.$$

Therefore

$$b = 15.5 \text{ ft. as a minimum.}$$

Example II. Fig. 445.—If $b = 15.5$ ft., what is the intensity of stress at D [Eq. (38)]? *Ans.* 3600 lb./sq. ft.

Example III.—The result of Example II may be verified as follows: A (of CD) $= b \times 1 = 15.5$ sq. ft., $e' = 8$ ft.,

$$e = \frac{b}{6}.$$

$$M_O = Qe' - Pe = 18{,}000 \times 8 - \frac{1800b^2}{6} = 144{,}000 - 300b^2 \text{ ft.-lb.}$$

$$\frac{I}{c} \text{ (of } CD \text{ with respect to } O \text{ as axis)} = \frac{1}{6} \times 1 \times b^2 = \frac{240}{6} = 40 \text{ ft.}^3$$

$$S_D = \frac{P}{A} + \frac{Mc}{I} = \frac{1800b}{b} + \frac{144{,}000 - 300b^2}{40}.$$

Or, since $b = 15.5$ ft.

$$S_D = 1800 + 1800 = 3600 \text{ lb./sq. ft.} = 25 \text{ lb./sq. in.}$$

Note.—The value of 3600 lb./sq. ft. (or 25 lb./sq. in.) will evidently be amply safe for a rock foundation. For earth, gravel, clay, etc., this value might exceed the allowable stress for such materials.

287. Eccentric Load Not in a Plane of Symmetry.—Figure 446 represents a short prism eccentrically loaded, the load P not acting in a plane of symmetry. For convenience, assume the prism rectangular. Take the axes of symmetry of the section CD as the X- and Y-axes and OO', the axis of the prism, as the Z-axis. (In general, X and Y must be principal axes, Art. 155.) Let e_x and e_y, respectively, be the x- and y-coordinates of the point of application of P, and c_x and c_y be the x- and y-coordinates of some point N in the section CD.

At O', the centroid of the end section, introduce two equal and opposite forces P_1 and P_2, each equal (and parallel) to P;

The load P is thus replaced by an axial load $P_1(= P)$ and a couple whose moment is $M = Pe$. The couple may be replaced by its x- and y-components.* That is, M may be replaced by M_x and M_y, where

$$M_x = Pe_y \qquad \text{and} \qquad M_y = Pe_x.$$

The stress at a point N in the section CD may be considered as composed of three stresses.

1. The direct stress, where A = area of section,

$$S_1 = \frac{P}{A}. \tag{a}$$

2. That due to the bending moment M_x [see Art. 162, Eq. (6)].

$$S_2 = \frac{M_x c_y}{I_x} = \frac{Pe_y c_y}{I_x}. \tag{b}$$

Note that X is an axis of symmetry and that therefore the flexure formula applies.

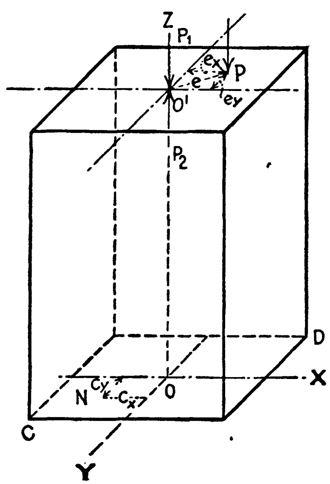


FIG. 446.

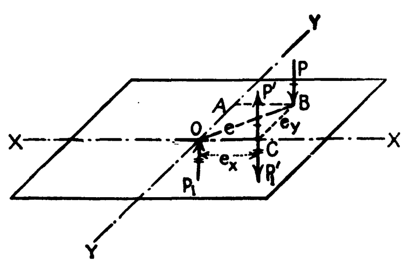


FIG. 447.

3. That due to the bending moment M_y,

$$S_3 = \frac{M_y c_x}{I_y} = \frac{Pe_x c_x}{I_y}. \tag{c}$$

* Given the couple PP_1 whose moment is $M = Pe$ (Fig. 447). Construct the rectangle $OABC$. At C introduce two equal and opposite forces P' and P_1' each equal and parallel to P. P and P' form a couple whose moment is $M_x = Pe_y$, and P_1 and P_1' form a couple whose moment is $M_y = Pe_x$.

Note that Y is an axis of symmetry.

The total stress at N is, therefore,

$$S = S_1 + S_2 + S_3 = \frac{P}{A} + \frac{Pe_yc_y}{I_x} + \frac{Pe_xc_x}{I_y}. \tag{39}$$

When Eq. (39) is used, signs must be considered. That is, e_x, e_y, c_x, and c_y are coordinates of points and, therefore, have sign. If the quadrant in which P acts is taken as the first quadrant, e_x and e_y are positive, but the signs of c_x and c_y, the coordinates of the point N at which the resultant stress is desired, depend upon the quadrant in which N lies. This will now be illustrated.

Example. Fig. 448.—A short post, 8 by 12 in., carries an eccentric load of $P = 48{,}000$ lb. as shown. Find the stress at each of the four corners of the section CD.

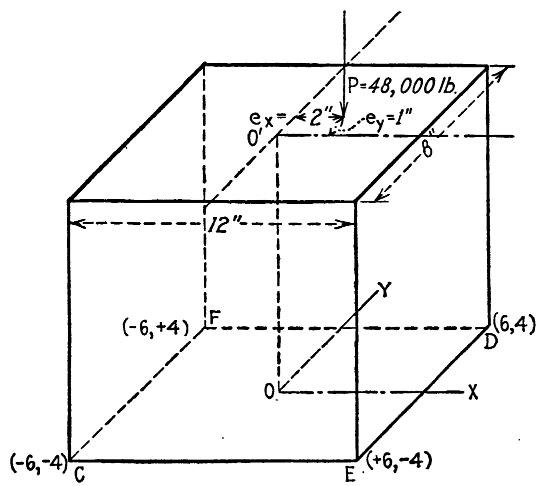


FIG. 448.

Taking XOY as the first quadrant, $e_x = 2$ in. and $e_y = 1$ in. At D, $c_x = 6$ in. and $c_y = 4$ in.

$$S_1 = \frac{P}{A} = \frac{48{,}000}{96} = 500 \text{ lb./sq. in.}$$

$$S_2 = \frac{Pe_yc_y}{I_x} = \frac{48{,}000 \times 1 \times 4}{\frac{1}{12} \times 12 \times 8^3} = 375 \text{ lb./sq. in.}$$

$$S_3 = \frac{Pe_xc_x}{I_y} = \frac{48{,}000 \times 2 \times 6}{\frac{1}{12} \times 8 \times 12^3} = 500 \text{ lb./sq. in.}$$

so that

$$S_D = 500 + 375 + 500 = 1375 \text{ lb./sq. in. (compression).}$$

At E, $c_x = +6$, and $c_y = -4$.

$$S_E = 500 - 375 + 500 = 625 \text{ lb./sq. in. (compression).}$$

At C, $c_x = -6$ and $c_y = -4$,

$$S_C = 500 - 375 - 500 = -375 \text{ lb./sq. in. (tension).}$$

At F, $c_x = -6$ and $c_y = +4$,

$$S_F = 500 + 375 - 500 = 375 \text{ lb./sq. in. (compression).}$$

PROBLEMS

258. In a laboratory test a steel rod, 1 in. in diameter and 30 in. long, is used as a round-ended column. All precautions are taken to make this column act as an ideal column. Referring to Fig. 419, first use the experimental curve T, and then Rankine's curve R, and find the load for which the column may be expected to fail. Referring to Fig. 420, find the breaking load by means of the straight line formula (curve S).

Ans. 16,100 lb.; 15,100 lb.; 15,700 lb.

Note.—In the following problems it is assumed that the columns are commercial columns and that, therefore, the formulas of Art. 274 should be used.

259. A steel rod, 1 in. in diameter and 45 in. long, is used as a prop. Use Fig. 422 and a factor of safety of 2.5. Find the safe load the prop may carry. Use Rankine's curve R. *Ans.* 5350 lb.

260. With a factor of safety of about 3, Eq. (27) may be used for a good grade of timber. Use a factor of safety of 4 and find the safe load a 10- by 10-in. wooden column, 15 ft. long, may carry. *Ans.* 55,500 lb.

261. A square wooden column of oak [Eq. (28)] is to be 18 ft. long and is to carry a load of $P = 117{,}600$ lb. Design the column.

Ans. 12 by 12 in.; $L/d = 18 < 25$.

262. A 5- by 5- by $\frac{7}{16}$-in. steel angle, 16 ft. long, is used as a column unsupported laterally. Find the buckling load.

Ans. 54,300 lb.; $L/k = 196 < 225$.

263. A cast-iron pipe, 14 ft. long, is used as a column. The pipe has an outside diameter of 9 in. and an inside diameter of 7 in. What safe load can the pipe carry?

264. Three 4- by 12-in. planks are firmly nailed together to form a square column 12 ft. long. Find the safe load this column can carry.

Ans. 123,800 lb.; $L/d = 12 < 25$.

265. In Problem 264, if the nails are rusted away or loosened so that the three planks act as separate units, find the total load P they can carry. First find L/d for a plank (see Art. 272). *Ans.* $L/d = 36$; $P = 42{,}600$ lb.

266. Find the maximum allowable length of a 4- by 4-in. wooden post that is to carry a load of (*a*) 14,400 lb., (*b*) 12,800 lb., (*c*) 6400 lb.

Ans. $3\frac{1}{3}$ ft.; 5 ft.; $10\frac{1}{3}$ ft.

267. Formula (5), Art. 261, gives the breaking load for a rounded-ended column. From it derive the formula for an ideal fixed-ended column.

$$Ans.\quad \frac{P}{A} = \frac{50{,}000}{1 + \dfrac{1}{36{,}000}\left(\dfrac{L}{K}\right)^2}.$$

268. A rectangular steel bar, 8 ft. long, is to be used as a column, pin ended one way and fixed ended the other way (Fig. 405*d*). The bar is carefully selected and the pins are carefully centered and are oiled so as to reduce friction to a minimum. The bar is 3 in. thick. Treat the bar as an ideal column, round ended one way and fixed ended the other way, and find the width for which the column is equally strong both ways (see Problem 267). What is the breaking load? *Ans.* 6 in.; 670,000 lb.

269. Two 12-in. 30-lb. channels are latticed together to form a column. Find the distance between channels, back to back with flanges turned out so that the column will be equally strong in all directions.

Ans. a = 7.15 in.

270. A 10-in. 40-lb. standard I-beam, 18 ft. long, carries a uniformly distributed load of 12,000 lb. and an axial load P. Determine the maximum safe value P may have. Use Eq. (22). Take 15,000 lb./sq. in. as the safe flexural stress (see Art. 279). *Ans.* 52,500 lb.

271. A wooden member, 16 ft. long, is to carry a concentrated transverse load of 4000 lb. at the middle (parallel to h) and an axial load of P = 4000 lb. Take 900 lb./sq. in. as the safe flexural stress. Let b = about $\frac{3}{4}h$. Design the member for combined bending and column action (see Art. 279).

Ans. 8 by 14 in.

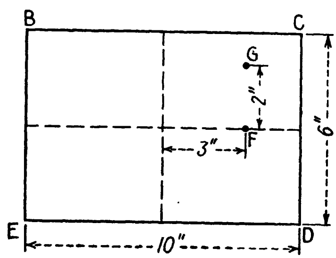


FIG. 449.

272. Figure 449 shows the section of a short wooden prism. What load may be applied at F, if the allowable fiber stress is 1000 lb./sq. in.?

273. What load may be applied at G? (Fig. 447, Problem 272.)

REVIEW PROBLEMS

274. A vertical plate 3 ft. wide and 2 ft. high covers an opening in a dam. The upper edge of the plate is 10 ft. below the water surface. Construct the stress solid for the plate. Compute the total water pressure on the plate. Compute the distance above the bottom of the plate at which the total pressure acts.

275. Figure 1 is the bottom of a column which supports 400 tons. The channels are steel, A is stone, and B is concrete which rests upon earth. The allowable bearing stresses are as follows: steel 20,000 lb./sq. in., stone 1000 lb./sq. in., concrete 400 lb./sq. in., and earth 3 tons/sq. ft. Compute the areas required for bearing.

276. A total load of 35 tons is supported by a 7-in. solid, circular column which rests upon a square, cast-iron bearing plate EF (S_c = 8500 lb./sq. in., S_s = 2200 lb./sq. in.). EF rests upon a square stone block AD (S_c = 300 lb./sq. in., S_s = 70 lb./sq. in.), which rests on earth (S_c = 2 tons/sq. ft.). Considering only the 35 tons, compute the side and thickness of EF and AD.

277. A $\frac{3}{8}$-in. round bolt has a square head $\frac{9}{16}$ in. on a side and $\frac{1}{4}$ in. thick. The bolt is in a $\frac{1}{2}$-in. round hole. A tension of 1800 lb. is applied to the bolt. Compute (*a*) unit stress in the shank, (*b*) unit bearing stress on the head, (*c*) unit shearing stress in the head.

278. Design a square-headed bolt to carry a tension of 30,000 lb. The hole is $\frac{1}{8}$ in. larger than the diameter of the shank. Bolt diameters to vary by $\frac{1}{8}$ in. from $\frac{1}{4}$ to 2 in., other dimensions by $\frac{1}{16}$ in. S_t = 16,000 lb./sq. in., S_s = 10,000 lb./sq. in., S_c = 20,000 lb./sq. in.

279. Compute the safe load for the bolt of Problem 278, using the commercial dimensions.

280. A vertical steel wire $\frac{3}{4}$ in. in diameter and 1500 ft. long supports a load of 6000 lb. Compute the total elongation (*a*) neglecting its own weight, (*b*) considering its own weight.

281. A vertical steel wire, $\frac{1}{2}$ in. in diameter and 1600 ft. long, has a $\frac{3}{8}$-in. steel wire 2400 ft. long attached to its lower end. The $\frac{3}{8}$-in. wire, in turn, supports 1500 lb. at its lower end. Compute the total elongation (*a*) neglecting weights of rods, (*b*) considering weights of rods.

282. A wire is composed of a steel core $\frac{1}{2}$ in. in diameter, surrounded by a layer of copper $\frac{1}{16}$ in. thick. Total length is 600 ft., and the total pull on the wire is 3000 lb. Assume the elastic limit is not exceeded. Compute (*a*) unit stress in each material, (*b*) total pull on each material, (*c*) unit elongation, (*d*) total elongation. E_s = 30,000,000 lb./sq. in., E_{cu} = 15,000,000 lb./sq. in.

283. A wrought-iron bar 4 in. wide and $\frac{3}{8}$ in. thick is placed between two steel bars, each 3 in. wide and $\frac{1}{2}$ in. thick. The total elongation is

0.018 in. in a gage length of 36 in. Compute (*a*) total pull on the combination, (*b*) unit stress in each material, (*c*) total load on each material. E_s = 29,000,000 lb./sq. in., E_{wi} = 27,000,000 lb./sq. in.

284. A wood block 10 by 10 in. is reinforced with a 5- by 5- by $\frac{3}{8}$-in. structural steel angle at each corner. Compute (*a*) total load that will cause a stress of 900 lb./sq. in. in the wood, (*b*) unit stress in the steel, (*c*) unit shortening if the original length is 18 in. E_s = 29,000,000 lb./sq. in., E_w = 1,800,000 lb./sq. in.

285. In Fig. 65, the length of each of the vertical rods is 50 ft. Rods B_1 are of steel and $\frac{1}{2}$ in. diameter (E = 29,000,000 lb./sq. in.). B_2 is of copper and 1 in. in diameter (E = 8,000,000 lb./sq. in.). The concentrated loads are each 4000 lb. Compute (*a*) the total tension in each rod, (*b*) unit stress in each rod, (*c*) total elongation.

286. Two $\frac{7}{16}$- by 6-in. plates are riveted together to make a lap joint. The four 1-in. rivets are one in the first and third rows and two in the second. Compute the safe load and efficiency of the joint, using unit stresses of Table II, page 107. S_t = 16,000 lb./sq. in.

287. Two $\frac{5}{16}$- by 6-in. plates are riveted together to make a lap joint. Use the arrangement of the four rivets as in Problem 286 and Table I, page 107. Compute the safe load and efficiency.

288. Two $\frac{3}{8}$- by 7-in. plates are riveted to make a lap joint. The nine $\frac{3}{4}$-in. rivets are arranged as follows: one in the first and fifth rows, two in the second and fourth, and three in the third row. Using the unit stresses of Table II, page 107, compute the safe load and efficiency. S_t = 16,000 lb./sq. in.

289. Two $\frac{1}{4}$- by 7-in. plates are riveted to make a lap joint. Arrange the nine $\frac{3}{4}$-in. rivets as in Problem 288. Using Table I, page 107, compute the safe load and efficiency.

290. Two $\frac{1}{2}$- by 4-in. plates are riveted to make a double-strap butt joint. The three $\frac{5}{8}$-in. rivets in one plate are arranged one in the first row and two in the second. Using Table II, page 107, compute the safe load and efficiency. S_t = 16,000 lb./sq. in.

291. Two $\frac{3}{8}$- by 4-in. plates are riveted to make a double-strap butt joint. Arrange the three $\frac{5}{8}$-in. rivets as in Problem 290. Using Table I, page 107, compute safe load and efficiency.

292. Two $\frac{5}{8}$- by 7-in. plates are riveted to make a double-strap butt joint. The six $\frac{7}{8}$-in. rivets in one plate are arranged one in the first row, two in the second row, and three in the third row. Using Table II, page 107, compute the safe load and efficiency. S_t = 16,000 lb./sq. in.

293. Two $\frac{1}{2}$- by 7-in. plates are riveted to make a double-strap butt joint. Arrange the six $\frac{7}{8}$-in. rivets as in Problem 292. Use Table I, page 107, and compute the safe load and efficiency.

294. Design a structural lap joint of maximum efficiency to carry 40,000 lb. Use $\frac{3}{4}$-in. rivets and the unit stresses of Table II, page 107.

295. Design a structural lap joint of maximum efficiency to carry 25,000 lb. Use $\frac{3}{4}$-in. rivets and Table I, page 107.

296. Design a structural lap joint of maximum efficiency to carry 80,000 lb. Use $\frac{7}{8}$-in. rivets and Table I, page 107.

297. A joint in a roof truss is made by welding a 4- by 4- by $\frac{5}{16}$-in. angle to a $\frac{3}{8}$-in. gusset plate. Design the weld to develop 80 per cent of the gross section of the angle. S_t = 20,000 lb./sq. in.

298. A joint in a roof truss is made by welding a 3- by 3- by $\frac{1}{4}$-in. angle to a $\frac{5}{16}$-in. gusset plate. Design the weld to develop 80 per cent of the gross section of the angle. S_t = 20,000 lb./sq. in.

299. A solid circular shaft is 24 ft. long and carries a torque of 16 ft.-tons. Compute (*a*) diameter of shaft, (*b*) helix angle in degrees, (*c*) angle of torsion in degrees. S_s = 10,000 lb./sq. in., E_s = 10,000,000 lb./sq. in.

300. A hollow circular steel shaft carries a torque of 6 ft.-tons. The outside diameter is 6 in., and S_s = 10,000 lb./sq. in. Compute (*a*) thickness of shaft, (*b*) diameter of solid shaft to carry above torque, (*c*) per cent saving in material of the hollow compared with the solid shaft.

301. A solid circular steel shaft is to withstand an external moment of 9000 ft.-lb. and a twist of 2° in a 6-ft. length. Compute the diameter of the shaft. S_s = 6 tons/sq. in., E_s = 6000 tons/sq. in.

302. A simple beam 16 ft. long is supported at each end. A uniformly distributed load of 3200 lb. covers the left half of the beam. Four feet from the right end, a concentrated load of 3200 lb. is applied. Draw the shear and moment diagrams.

303. A simple beam 16 ft. long is supported at each end. A uniform load of 4800 lb. extends the full length of the beam. Four feet from the left end, a concentrated load of 2400 lb. is applied. Draw the shear and moment diagrams.

304. A beam 12 ft. long is simply supported at the right end and 2 ft. from the left end. At the left end, a concentrated load of 1200 lb. is applied. Over the right 8 ft. of the beam, a uniformly distributed load of 2560 lb. is applied. Draw the shear and moment diagrams.

305. A beam 48 ft. long is simply supported at the right end and 12 ft. from the left end. The left 30 ft. support a uniformly distributed load of 7200 lb., and 6 ft. from the right support, a concentrated load of 1800 lb. is applied. Compute (*a*) the shear just to the right of the left support, (*b*) the shear 20 ft. from the right end, (*c*) the moment 13 ft. from the left end, (*d*) the moment 19 ft. from the right end.

306. A beam 50 ft. long is simply supported at the left end and 10 ft. from the right end. A concentrated load of 900 lb. is applied 6 ft. from the left support. The right 30 ft. support a uniformly distributed load of 2700 lb. Draw shear and moment diagrams. What is the maximum shear and maximum moment?

307. A beam 15 ft. long is simply supported at the right end and 3 ft. from the left end. The left 3 ft. of the beam support a uniformly distributed load of 1800 lb. Two feet to the right of the left support is a concentrated load of 3600 lb. The right 6 ft. of the beam support a uniformly distributed load of 7200 lb. Draw the shear and moment diagrams. What is the maximum shear and the maximum moment?

308. A beam 22 ft. long is simply supported at the right end and 6 ft. from the left end. At the left end of the beam is a concentrated load of 1000 lb. The right 16 ft. support a uniformly distributed load of 4000 lb., and 14 ft. from the right end is a concentrated load of 1000 lb. Draw the shear

and moment diagrams. What is the maximum shear and the maximum moment?

309. A beam 40 ft. long supports a uniformly distributed load of 200 lb./linear foot. The supports are 8 ft. from each end of the beam. Draw shear and moment diagrams. What is the maximum shear and the maximum moment?

310. Solve Problem 309 if the supports are 9 ft. from each end of the beam.

311. A beam 18 ft. long is simply supported at each end. A triangular load of 9000 lb. extends from the left support to within 6 ft. of the right support, with the toe of the load at the right. Draw shear and moment diagrams and find maximum shear and maximum moment.

312. A beam 18 ft. long is simply supported at each end. A triangular load of 9000 lb. extends from the right support to within 6 ft. of the left support, with the toe of the load at the right support. Draw shear and moment diagrams and find maximum shear and maximum moment.

313. A beam 15 ft. long is simply supported at each end. A uniformly distributed load of 4800 lb. extends the entire length of the beam. On top of this load is a triangular load of 2400 lb. Compute the maximum moment.

314. A beam 24 ft. long is simply supported at the left end and 4 ft. from the right end. A triangular load of 6 tons extends over the left 12 ft. of the beam, with the toe of the load to the right. Ten feet from the right end is a concentrated load of 2 tons and another 2 tons at the right end. Draw the shear and moment diagrams, and find the maximum shear and the maximum moment.

315. A beam 24 ft. long is simply supported at the right end and 4 ft. from the left end. A uniformly distributed load of 6000 lb. extends over the left 20 ft. of the beam. On top of this load is a triangular load of 9000 lb. with the toe of the load to the left. Compute the two local maximum moments.

316. A 24-ft. simple beam is supported at the left end and 4 ft. from the right end. Six feet from the left end is a concentrated load of 2000 lb. Over the right 12 ft. is a uniformly distributed load of 3000 lb. and a triangular load of 6000 lb., with the toe at the right end of the beam. Compute the maximum moment.

317. A 16-ft. simple beam is supported at the right end and 4 ft. from the left end. Two feet from the right end is a concentrated load of 9000 lb. Over the left 10 ft. of the beam is a uniformly distributed load of 36,000 lb. If the rectangular wood beam is 8 in. wide, compute the minimum safe height. S_t = 1600 lb./sq. in.

318. A 30-ft. simple beam is supported at the right end and 6 ft. from the left end. The 6-ft. cantilever supports a uniformly distributed load of 1200 lb. Four feet to the right of the left support is a concentrated load of 2400 lb. The right 12 ft. support a uniformly distributed load of 4800 lb. If the rectangular wood beam is 4 in. wide, compute the minimum safe height. S_t = 1500 lb./sq. in.

319. A 30-ft. simple beam is supported 4 ft. from the left end and 6 ft. from the right end. A uniform load of 15,000 lb. is distributed over the 30 ft., and 12 ft. from the right end of the beam is a concentrated load of 350 lb. Design an economical wood beam ($3b = h$). S_t = 1000 lb./sq. in.

320. A 20-ft. simple beam is supported 5 ft. from the left end and 3 ft. from the right end. A uniformly distributed load of 32,000 lb. is distributed over the 20 ft. A concentrated load of 800 lb. is 9 ft. from the left end. Design an economical wood beam ($2b = h$). $S_t = 1200$ lb./sq. in.

321. A 15-ft. simple beam, supported 3 ft. from each end, carries a uniformly distributed load of 36,000 lb. Draw the shear and moment diagrams, and design an economical wood beam ($2b = h$). $S_t = 800$ lb./sq. in.

322. A 16-ft. simple beam, supported at the ends, carries a uniformly distributed load of 20 tons. Design an economical American Standard I-beam ($S = 10$ tons/sq. in.) (*a*) neglecting weight of beam, (*b*) considering weight of beam.

323. Solve Problem 322 if the 20 tons is concentrated at the center of the beam.

324. Solve Problem 322 if the 20 tons is concentrated 6 ft. from one end of the beam.

325. A 32-ft. simple beam is supported at the left end and 8 ft. from the right end. A load of 14,400 lb. is uniformly distributed over the left 16 ft. of the beam. A second load of 3600 lb. is uniformly distributed over the right 8 ft., and a concentrated load of 2160 lb. is 4 ft. to the left of the right support. $S_t = 10$ tons/sq. in. Considering the weight of the beam, design an economical (*a*) American Standard I-beam, (*b*) WF beam.

326. A 40-ft. simple beam is supported 6 ft. from the left end and 10 ft. from the right end. A load of 50,000 lb. is uniformly distributed over the 40 ft. and a concentrated load of 1250 lb. is 18 ft. from the right end of the beam. $S_t = 10$ tons/sq. in. Considering the weight of the beam, design an economical (*a*) American Standard I-beam, (*b*) WF beam.

327. A 40-ft. simple beam is supported at each end. A triangular load of 12 tons is distributed over 24 ft. with the toe of the load 4 ft. from the right support. $S_t = 10$ tons/sq. in. Considering the weight of the beam, design an economical (*a*) American Standard I-beam, (*b*) WF beam.

328. Two wheel loads, of 10 tons and 5 tons, respectively, are 15 ft. apart and roll across a 27-ft. span. Draw the dimension sketches, and compute the maximum moment under each wheel. $S_t = 10$ tons/sq. in. Considering its own weight, design an economical (*a*) American Standard I-beam, (*b*) WF beam.

329. Two wheel loads, of 3 tons and 2 tons, respectively, are 10 ft. apart and roll across a 30-ft. span. Draw dimension sketches, and compute the maximum moment under each wheel. $S_t = 6$ tons/sq. in. Considering its own weight, design an economical (*a*) American Standard I-beam, (*b*) WF beam.

330. An 8-ft. simple wood beam 4 by 16 in., supported at the ends, has a concentrated load of 5 tons at the center of the span. Compute the unit shearing stress at points 2, 4, and 6 in. from the bottom of the beam.

331. A 20-in. 70-lb. American Standard I-beam, 20 ft. long with supports at the ends, has a concentrated load of 20 tons 9 ft. from one end. Compute the maximum unit shearing stress by (*a*) the exact formula, (*b*) the approximate method.

332. A 26-ft. simple beam is supported at the left end and 6 ft. from the right end. A load of 4500 lb. is uniformly distributed over the left 10 ft.

One concentrated load of 900 lb. is 4 ft. to the left of the right support and a second 900 lb. is at the right end of the beam. S_t = 1040 lb./sq. in., S_s = 80 lb./sq. in. Design an economical wood beam of nominal commercial size.

333. A simple beam on end supports has a concentrated load P at its center. E and I are constants. From the equation of the elastic curve, find the slope of the curve and its deflection at a distance of $L/3$ from the end.

334. Solve Problem 333 with a uniformly distributed load $W = wL$ instead of the load P.

335. A cantilever beam has a concentrated load P at the free end. E and I are constants. From the equation of the elastic curve, find its slope and deflection at a distance of $2L/3$ from the support.

336. Solve Problem 335 with a uniformly distributed load $W = wL$ instead of the load P.

337. Find the deflection of the beam of Problem 333 by the Slope-deflection Method.

338. Find the deflection of the beam of Problem 334 by the Slope-deflection Method.

339. Solve Problem 333 by the Moment-area Method.

340. Find the deflection of the beam of Problem 333 by the Theorem of Three Moments Equation.

341. Find the deflection of the beam of Problem 334 by the Theorem of Three Moments Equation.

342. A simple beam on two end supports is loaded at the third points with two equal concentrated loads, P, P. E and I are constants. Find the maximum deflection by the Slope-deflection Method.

343. Solve Problem 342 by the Moment-area Method.

344. Solve Problem 342 by the Theorem of Three Moments Equation.

345. A continuous beam rests on three supports which are at the same level. Each panel is 15 ft. long. The two equal concentrated loads of 9 tons are 10 ft. from the center support. Compute sufficient data, draw shear and moment diagrams, and find maximum shear and maximum moment.

346. Solve Problem 345 if the concentrated loads are replaced by two 9-ton uniformly distributed loads extending from the free ends of the beam for 5 ft.

347. A continuous beam rests on three supports which are at the same level. The left panel is 18 ft. and the right panel 16 ft. A 4-ton load is uniformly distributed over the entire left panel. Six tons is concentrated 4 ft. from the right end. Compute sufficient data, draw the shear and moment diagrams, and find the maximum shear and maximum moment.

348. A continuous beam on three level supports is 36 ft. long. The left panel, 20 ft. long, has a uniformly distributed load of 9 tons on the left 6 ft. The right panel, 16 ft. long, has a concentrated load of 3 tons at 4 ft. from the right end. Compute sufficient data, draw the shear and moment diagrams, and find the maximum shear and maximum moment.

349. A continuous beam on three level supports is 31 ft. long. The left support is 3 ft. from the left end of the beam, the right support is at the right end of the beam and the other support is 16 ft. from the right end.

One ton is concentrated at the left end, 12 tons is concentrated 4 ft. from the left support, and 8 tons is uniformly distributed over the right panel. S_t = 1200 lb./sq. in. Design an economical wood beam ($2b = h$, approximately).

350. Considering its own weight, select an economical rolled beam to support the loads of Problem 345. S_t = 20,000 lb./sq. in. (*a*) American Standard I-beam, (*b*) WF beam.

351. Considering its own weight, select an economical rolled beam to support the loads of Problem 346. S_t = 20,000 lb./sq. in. (*a*) American Standard I-beam, (*b*) WF beam.

352. Using the equation of the Theorem of Three Moments, compute the deflection at the center of the 12-ft. panel for the beam of Problem 349. E = 1,000,000 lb./sq. in.

353. A continuous beam on four level supports is 40 ft. long. The two end panels are each 14 ft. long and support a uniformly distributed load of 12 tons over the 14 ft. The center panel supports a concentrated load of 6 tons at the center of the 12 ft. Compute sufficient data, draw the shear and moment diagrams, and find the maximum shear and maximum moment.

354. Compute the safe load for a 2- by 2- by $\frac{1}{4}$-in. angle, 4 ft. 10.5 in. long, used as a column. Use Eq. (24) or (25).

355. Compute the safe load for a 3- by $2\frac{1}{2}$- by $\frac{7}{16}$-in. angle, 6 ft. 06 in. long, used as a column. Use Eq. (24) or (25).

356. Compute the safe load for an 8- by 8-in. wood column 15 ft. long. Use Eq. (27).

357. Design an economical angle with unequal legs, 8 ft. 04 in. long, to support a safe load of 25,000 lb. [Eq. (24) or (25).]

358. Design an economical angle with unequal legs, 8 ft. 04 in. long, to support a safe load of 50,000 lb. [Eq. (24) or (25).]

359. Design an economical angle with equal legs for the data of Problem 357.

360. Design an economical angle with equal legs for the data of Problem 358.

361. Design an economical wood column of nominal commercial size, 15 ft. long, for a safe load of 35,000 lb. [Eq. (27).]

362. Design an economical wood column of nominal commercial size, 15 ft. long, for a safe load of 70,000 lb. [Eq. (27).]

363. Design an economical wood member of nominal commercial size, 20 ft. long, to carry an axial compression of 4000 lb. and an eccentric concentrated load of 2000 lb. at 8 ft. from one end. S_t = 1100 lb./sq. in. Eq. (27).

364. Design an economical wood member of nominal commercial size, 10 ft. long, to carry an axial compression of 20,000 lb. and an eccentric concentrated load of 5000 lb. at 3 ft. from one end. S_t = 1200 lb./sq. in. [Eq. (28).]

365. Design a pair of economical American Standard channels, 15 ft. long, to support an axial compression of 40,000 lb. and an eccentric concentrated load of 15,000 lb. at 6 ft. from one end. S_t = 20,000 lb./sq. in. [Eq. (24) or (25).]

366. A weight of 7000 lb. is suspended from a steel bar 0.60 in. in diameter. What unit stress is induced in the bar?

367. In Problem 366 what is the unit elongation if $E = 30(10)^6$ lb./sq. in.? If the rod was originally 500 ft. long, what is the total elongation?

368. A concrete pier is to support a load of 1,120,000 lb. uniformly distributed. The American Institute of Concrete Construction specifies that the allowable stress may be 0.4 of the ultimate. Compute the section area of the pier, using 3500 lb./sq. in. concrete.

If the stress in a member is 0.4 of the ultimate, what safety factor does this imply?

369. Solve Problem 275 if the column load is 240 tons.

370. A total load of 25 tons is supported by an 8-in. circular concrete column which rests upon a square cast-iron bearing plate *EF* ($S_c = 8000$ lb./sq. in., $S_s = 2000$ lb./sq. in.). *EF* rests on a square stone block *AD* ($S_c = 800$ lb./sq. in., $S_s = 150$ lb./sq. in.), which rests on earth ($S_c = 2.5$ tons/sq. ft.). Considering only the 25 tons, compute the side and thickness of *EF* and *AD*.

371. A $\frac{7}{8}$-in. round bolt has a square head $1\frac{5}{16}$ in. on a side and $\frac{9}{16}$ in. high. The bolt is in a $\frac{15}{16}$-in. hole. If a tension of 8300 lb. is applied to the bolt, compute (*a*) unit tensile stress in the shank, (*b*) unit bearing stress on the head, (*c*) unit shearing stress in the head.

372. A $1\frac{1}{8}$-in. round bolt has a square head $1\frac{11}{16}$ by $\frac{3}{4}$ in. The bolt is in a $1\frac{3}{16}$-in. hole. If a tension of 16,000 lb. is applied to the bolt, compute (*a*) unit tensile stress in the shank, (*b*) unit bearing stress on the head, (*c*) unit shearing stress in the head.

373. Compute the length of a brass rod that will be on the point of rupturing in tension under its own weight when hung vertically. Brass weighs 534 lb./cu. ft. Ultimate stress is 50,000 lb./sq. in.

374. A steel bar 1 by $\frac{3}{4}$ in. sustains a pull of 42,000 lb. What are the induced normal and shearing stresses per unit area on sections making angles of 40°, 45°, and 50° with the cross-section?

375. A vertical plate, 2 ft. wide and 4 ft. high, covers an opening in a dam. The upper edge of the plate is 8 ft. below the water surface. Construct the stress solid for the plate. Compute (*a*) the total water pressure on the plate, (*b*) the distance below the water surface at which the resultant pressure acts.

376. Design a square-headed bolt to sustain a tension of 50,000 lb. The hole is $\frac{1}{16}$ in. larger than the shank. $S_t = 20{,}000$ lb./sq. in., $S_s = 10{,}000$ lb./sq. in., $S_c = 20{,}000$ lb./sq. in.

377. Compute the safe load for the bolt of the preceding problem, using the commercial dimensions.

378. A vertical steel wire $\frac{3}{8}$ in. in diameter and 3,000 ft. long supports a tension of 800 lb. Compute the total elongation (*a*) neglecting its own weight, (*b*) considering its own weight.

379. A vertical steel wire $\frac{1}{2}$ in. in diameter and 2000 ft. long supports a tension of 3000 lb. Compute the total elongation (*a*) neglecting its own weight, (*b*) considering its own weight.

380. A vertical steel wire $\frac{7}{16}$ in. in diameter and 1800 ft. long has a $\frac{5}{16}$-in. wire 2600 ft. long attached to its lower end. The $\frac{5}{16}$-in. wire, in turn, supports 1000 lb. at its lower end. Compute the total elongation due to the 1000-lb. weight only.

381. In the preceding problem assume that the 1000-lb. weight is removed. Compute the total elongation due to the weights of the rods.

382. In Problem 380, considering both the end load and the weights of the rods, compute the maximum unit tensile stress induced in the rod.

383. A steel wire is $\frac{9}{16}$ in. in diameter. It is 400 ft. long when subjected to a pull of 1500 lb. What will be its length when it is subjected to a pull of 7500 lb.?

384. A steel bar $\frac{3}{8}$ in. square and 20 ft. long is subjected to a pull of 2200 lb., and the ends are then fastened to immovable supports. Compute the temperature rise for the stress in the bar to be zero.

385. A steel rod 18 ft. long is rigidly held at its ends. If, at 30°F., the tension in the rod is 12,000 lb./sq. in., what will be the stress at 90°F.?

386. A steel rod 14 ft. long is rigidly held at its ends. If, at 100°F., the tension in the rod is 160 lb./sq. in., what will be the stress at 25°F.?

387. A block of wood 8 in. square is reinforced with a 3- by 3- by $\frac{3}{8}$-in. structural steel angle at each corner. Compute the (*a*) total compression that will cause a stress of 1000 lb./sq. in. in the wood, (*b*) the unit stress in the steel. Area of one angle is 2.11 sq. in. $E_s = 30(10)^6$ lb./sq. in., $E_w = 15(10)^5$ lb./sq. in.

388. A wood block 4 in. square has two 4- by $\frac{1}{4}$-in. steel plates bolted to opposite sides. The compound member is subjected to an axial compression of 35,000 lb. Compute the unit stress in each material. See preceding problem for moduli of elasticity.

389. Two $\frac{5}{16}$ by 8-in. steel plates are riveted together to make a lap joint. The nine $\frac{7}{8}$-in. rivets are arranged three in each row. Compute the safe load and efficiency of the joint, using the unit stresses of Table I, page 107.

390. Solve the preceding problem if the nine rivets are arranged one in the first and fifth rows, two in the second and fourth rows, and three in the third row.

391. Two $\frac{1}{2}$ by $11\frac{1}{4}$-in. steel plates are riveted together to make a lap joint. The nine $1\frac{1}{4}$-in. rivets are arranged three in each row. Compute the safe load and efficiency. See Table I, page 107.

392. Solve the preceding problem, but arrange the $1\frac{1}{4}$-in. rivets as in Problem 390.

393. Two $\frac{1}{2}$- by 8-in. steel plates are riveted together to make a double-strap butt joint. The six $\frac{7}{8}$-in. rivets are arranged three in each row in one main plate. The rectangular cover plates are $\frac{5}{16}$ in. thick. Compute the safe load and efficiency of the joint. (Table I, page 107.)

394. Solve the preceding problem if the six rivets are arranged one in the first row, two in the second, and three in the third row.

395. Two $\frac{3}{4}$- by $11\frac{1}{4}$-in. steel plates are riveted together to make a double-strap butt joint. The six $1\frac{1}{4}$-in. rivets in one side of the joint are arranged three in each row. Cover plates are $\frac{7}{16}$ in. thick. Compute the safe load and efficiency of the joint. (Table I, page 107.)

396. Solve the preceding problem if the $1\frac{1}{4}$-in. rivets are arranged as in Problem 394.

397. Design a structural lap joint of maximum efficiency to carry 39,000 lb. Use $\frac{3}{4}$-in. rivets and Table I, page 107.

398. Design a structural lap joint of maximum efficiency to carry 54,000 lb. Use $\frac{7}{8}$-in. rivets and Table I, page 107.

399. A 5- by 5- by $\frac{7}{16}$-in. angle is to be welded to a $\frac{1}{2}$-in. gusset plate. Design the weld to develop 80 per cent of the gross section of the angle. S_t = 10 tons/sq. in.

400. A 6- by 6- by $\frac{1}{2}$-in. angle is to be welded to a $\frac{9}{16}$-in. gusset plate. Design the weld to develop 80 per cent of the gross section of the angle. S_t = 10 tons/sq. in.

401. Design a solid circular shaft to resist a torque of 6000 ft.-lb. S_s = 12,000 lb./sq. in.

402. If the shaft of the preceding problem is 24 ft. long, compute the angle of twist. $E_s = 12(10)^6$ lb./sq. in.

403. Design a solid circular shaft to resist a torque of 8000 ft.-lb. S_s = 12,000 lb./sq. in.

404. If the shaft of the preceding problem is 30 ft. long, compute the angle of twist.

405. A steel rod $\frac{3}{4}$ in. in diameter is to be twisted through an angle of 90°. Compute the minimum length of the rod. S_s = 12,000 lb./sq. in.

406. A steel shaft 5 in. in diameter is to transmit 1500 hp. with an allowable stress of 12,000 lb./sq. in. Compute the r.p.m.

407. A steel shaft 4 in. in diameter rotates at 210 r.p.m. Compute the horsepower transmitted if S_s = 12,000 lb./sq. in.

408. Compute the outside diameter of a hollow steel shaft to resist a torque of 10 ft.-tons. Assume inside diameter 0.8 of the outside diameter. S_s = 10,000 lb./sq. in.

409. Compute the outside diameter of a hollow steel shaft to transmit 6,000 hp. at 120 r.p.m. Assume inside diameter 0.6 of the outside diameter. S_s = 8,000 lb./sq. in.

410. A hollow steel shaft is to transmit a torque of 12 ft.-tons. If the outside diameter is 6 in., compute the thickness. S_s = 10,000 lb./sq. in.

411. An 18-ft. simple beam on end supports has a concentrated load of 9 tons 5 ft. from the left end and a concentrated load of 6 tons 8 ft. from the right end. Compute the vertical shear and the bending moment (*a*) 7 ft. from the left end, (*b*) 9 ft. from the right end.

412. Solve the preceding problem if the beam has an additional uniform load of 27 tons distributed over the 18 ft.

413. Solve Problem 412 if the right support is 6 ft. from the right end of the beam.

414. Solve Problem 412 if the left support is 6 ft. from the left end of the beam.

415. A 20-ft. simple beam on end supports has a concentrated load of 8 tons 5 ft. from the left end and a concentrated load of 6 tons 8 ft. from the right end. Compute the vertical shear and the bending moment (*a*) 6 ft. from the left end, (*b*) 10 ft. from the right end.

416. Solve the preceding problem if the beam has an additional uniform load of 16 tons distributed over the 20 ft.

417. Solve the preceding problem if the left support is 6 ft. from the left end.

418. Solve Problem 416 if the right support is 9 ft. from the right end.

419. A 20-ft. simple beam is supported at the left end and 4 ft. from the right end. At the right end is a concentrated load of 2 tons. Beginning at the left support and extending 12 ft. to the right is a uniformly distributed load of 18 tons. Compute (*a*) the shear 6 ft. from the right end of the beam and 8 ft. from the left end, (*b*) the moment 8 ft. from the right end and 6 ft. from the left end.

420. A 30-ft. simple beam rests on end supports. A uniform load of 24 tons is distributed over the 30 ft., and a concentrated load of 6 tons is 5 ft. from the left end. Compute the maximum shear and the maximum moment.

421. Solve the preceding problem if the right support is 6 ft. from the end.

422. In Problem 420 an additional load of 5 tons is placed 4 ft. from the right end. Compute the maximum shear and maximum moment.

423. In Problem 421 place a 4-ton load at the outer end of the cantilever. Compute the maximum shear and maximum moment.

424. A 16-ft. beam is supported at each end. A uniform load of 2400 lb. is distributed over the right half of the beam. Four feet from the left end is a concentrated load of 2400 lb. Draw the shear and moment diagrams, and compute the maximum moment.

425. A 16-ft. beam rests on end supports. A uniform load of 3200 lb. extends the full length of the beam. Four feet from the right end is a concentrated load of 1600 lb. Draw the shear and moment diagrams.

426. A 12-ft. beam is simply supported at the left end and 2 ft. from the right end. At the right end is a concentrated load of 1800 lb. Extending 8 ft. from the left end is a uniformly distributed load of 3840 lb. Draw the shear and moment diagrams.

427. A 48-ft. beam is simply supported at the left end and 12 ft. from the right end. On the right 30 ft. is a uniformly distributed load of 3600 lb. A concentrated load of 900 lb. is 6 ft. from the left end. Compute (*a*) the shear 13 ft. from the right end and 18 ft. from the left end, (*b*) the moment 18 ft. from the right end and 16 ft. from the left end.

428. A 23-ft. beam is supported 4 ft. from the left end and 3 ft. from the right end. The left 20 ft. has a uniformly distributed load of 20 tons. At the right end is an 8-ton load. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

429. A 50-ft. beam is simply supported at the right end and 10 ft. from the left end. A concentrated load of 2700 lb. is 6 ft. from the right end. On the left 30 ft. is a uniformly distributed load of 8100 lb. Draw the shear and moment diagrams, and determine the maximum shear and maximum moment.

430. A 23-ft. beam is supported 3 ft. from the left end and 4 ft. from the right end. At the left end is a concentrated load of 6 tons. Over the right 4 ft. is a uniformly distributed load of 3 tons. Over the 16 ft. between supports is a uniformly distributed load of 24 tons. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

431. A 15-ft. beam is simply supported at the left end and 3 ft. from the right end. On the cantilever is a uniformly distributed load of 2700 lb.

Two feet to the left of the right support is a concentrated load of 5400 lb. The left 6 ft. support a uniformly distributed load of 10,800 lb. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

432. A 22-ft. beam is simply supported at the left end and 6 ft. from the right end. The 16 ft. between supports has a uniformly distributed load of 6000 lb. Fourteen feet from the left end is a concentrated load of 1500 lb. At the right end is a concentrated load of 1500 lb. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

433. A 25-ft. beam has a uniformly distributed load of 300 lb./ft. The supports are 5 ft. from each end. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

434. Solve the preceding problem if the supports are 6 ft. from the ends.

435. Solve Problem 433 if the supports are 4 ft. from the ends. Which of these three problems has the least maximum moment?

436. An 18-ft. beam is simply supported at each end. A triangular load of 12,000 lb. extends from the left support for 12 ft. with the toe of the load to the right. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

437. An 18-ft. beam is simply supported at each end. A triangular load of 6 tons extends from the right support over 12 ft. with the toe of the load at the right support. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

438. A 15-ft. beam is simply supported at each end. A uniformly distributed load of 6000 lb. covers the 15 ft. On top of this load is a triangular load of 3000 lb. Compute the maximum moment.

439. A 20-ft. beam on end supports is loaded with a uniform load of 15 tons, on top of which is a triangular load of 12 tons. Compute the maximum moment.

440. A 24-ft. beam is simply supported at the right end and 4 ft. from the left end. A triangular load of 9 tons extends over the right 12 ft. with the toe of the load to the left. Ten feet from the left end is a concentrated load of 3 tons and another 3 tons at the left end. Compute the maximum moment.

441. A 24-ft. beam is simply supported at the left end and 4 ft. from the right end. A uniform load of 9000 lb. extends over the right 20 ft. On top of this load is a triangular load of 13,500 lb. with the toe of the load at the right end of the beam. Compute the maximum moment.

442. If 10 tons/sq. in. is the maximum allowable fiber stress in an I-beam, compute the maximum bending moment a 12-in. 45-lb. I-beam can carry. $I = 284.1$ in.^{4}

443. A 12-ft. rectangular wood beam on end supports has a central concentrated load of 3200 lb. If the depth of the beam is 12 in., compute its width. $S = 1200$ lb./sq. in.

444. A 12-ft. rectangular wood beam on end supports is 12 in. high. It is loaded with a central concentrated load of 3200 lb. and a uniform load of 6400 lb. Compute its width. $S = 1200$ lb./sq. in.

445. A 16-ft. beam is simply supported at the left end and 4 ft. from the right end. Two feet from the left end is a concentrated load of 4500 lb. On the right 10 ft. is a uniform load of 18,000 lb. Compute the minimum safe height if the wood beam is 6 in. wide. $S = 1000$ lb./sq. in.

446. A 30-ft. simple beam is supported at the left end and 6 ft. from the right end. The cantilever has a uniform load of 900 lb. Four feet to the left of the right support is a concentrated load of 1800 lb. The left 12 ft. has a uniform load of 3600 lb. If the rectangular wood beam is 3 in. wide, compute the minimum safe height. $S = 1600$ lb./sq. in.

447. A 30-ft. simple beam is supported 6 ft. from the left end and 4 ft. from the right end. A uniform load of 9000 lb. is distributed over the 30 ft. Twelve feet from the left end is a concentrated load of 210 lb. Design an economical wood beam ($3b = h$). $S = 1200$ lb./sq. in.

448. A 20-ft. simple beam is supported 5 ft. from the right end and 3 ft. from the left end. A uniform load of 20,000 lb. is distributed over the 20 ft. Nine feet from the right end is a concentrated load of 500 lb. Design an economical wood beam ($2b = h$ approx.). $S = 1400$ lb./sq. in.

449. A 16-ft. simple beam, supported 3 ft. from each end, has a uniform load of 48,000 lb. Draw the shear and moment diagrams, and design an economical wood beam ($2b = h$ approx.). $S = 1000$ lb./sq. in.

450. Solve the preceding problem if the supports are 4 ft. from each end.

451. Design an economical wood beam for the loads of Problem 447. Given $S = 1200$ lb./sq. in., $S_s = 100$ lb./sq. in.

452. Design an economical wood beam for the loadings of Problem 448. Given $S = 1400$ lb./sq. in., $S_s = 84$ lb./sq. in.

453. Design an economical wood beam for the loadings of Problem 449. Given $S = 1000$ lb./sq. in., $S_s = 90$ lb./sq. in.

454. An 18-ft. beam on end supports has a concentrated load of 12 tons placed 8 ft. from the right support. Select an economical American Standard I-beam. $S = 10$ tons/sq. in.

455. In the preceding problem check the design, considering the weight of the beam.

456. An 18-ft. simple beam on end supports has a uniform load of 12 tons over the span. Select an economical American Standard I-beam (*a*) neglecting weight of beam, (*b*) considering weight of beam. $S = 10$ tons/sq. in.

457. Solve the preceding problem if the 12 tons is concentrated at the center of the span.

458. Solve Problem 456 if the 12 tons is concentrated 6 ft. from one end.

459. An 18-ft. simple beam on end supports has two 6-ton loads at the third points. Design an economical American Standard I-beam (*a*) neglecting weight of beam, (*b*) considering weight of beam. $S = 10$ tons/sq. in.

460. Solve Problem 459 if the loads are placed at the quarter points.

461. An 18-ft. beam is simply supported at the ends. Six tons is concentrated 5 ft. from the left end and 9 tons at 7 ft. from the right end. Select an economical American Standard I-beam (*a*) neglecting weight of beam, (*b*) considering weight of beam. $S = 10$ tons/sq. in.

462. A 20-ft. wooden joist is $2\frac{1}{2}$ in. wide. It supports a uniform load of 70 lb./ft. and a concentrated load of 800 lb. at 4 ft. from the right end. Compute the commercial size of the joist. $S = 1200$ lb./sq. in.

463. Compute the spacing of 2- by 8-in. by 16-ft. joists to carry a floor load of 45 lb./sq. ft. of floor area. $S = 1600$ lb./sq. in.

464. A 32-ft. simple beam is supported at the right end and 8 ft. from the left end. A uniform load of 12,000 lb. is distributed over the right 16 ft. of the beam. A second uniform load of 3000 lb. is distributed over the left 8 ft. A concentrated load of 1800 lb. is 4 ft. to the right of the left support. $S = 10$ tons/sq. in. Considering its own weight design (*a*) American Standard I-beam, (*b*) W̅F beam.

465. An 18-ft. beam is supported 3 ft. from one end and 4 ft. from the other end. A uniform load of 9 tons is distributed over the 18 ft. Design a standard I-beam, considering its own weight. $S = 10$ tons/sq. in.

466. A 15-in. 65-lb. American Standard I-beam, 40 ft. long, has a section modulus of 84.3 in.^{3} and rests on end supports. Neglect weight of beam. It is to support a uniform load over 16 ft. of the span beginning 8 ft. from the left support. (*a*) Locate the point of maximum moment, (*b*) compute the maximum moment in terms of W, (*c*) compute W if $S = 10$ tons/sq. in.

467. Considering the weight of the beam, compute W for the data of the preceding problem.

468. A 15-in. 65-lb. standard I-beam on end supports is to carry a warehouse load of 600 lb./ft. of length. Considering its own weight, compute the maximum safe length of the beam. $S = 9$ tons/sq. in., $S_s = 6$ tons/sq. in.

469. A 30-ft. simple beam is supported at each end. A triangular load of 10 tons is distributed over 18 ft. with the toe of the load 5 ft. from the right support. $S = 10$ tons/sq. in. Considering its own weight, design an economical (*a*) American Standard I-beam, (*b*) W̅F beam.

470. A 24-ft. simple beam is supported at the left end and 4 ft. from the right end. A uniform load of 12 tons is distributed over the entire beam. On top of this load is a triangular load of 9 tons with the toe of the load at the right end of the beam. $S = 10$ tons/sq. in. Neglecting the weight of the beam, design an economical (*a*) American Standard I-beam, (*b*) W̅F beam.

471. Solve the preceding problem, considering the weight of the beam.

472. Two wheel loads of 4 tons and 6 tons, respectively, are 12 ft. apart and roll across a span of 20 ft. End supports. Draw the dimension sketches and compute the maximum moment under each wheel. $S = 9$ tons/sq in. Considering its own weight, design an economical (*a*) American Standard I-beam, (*b*) W̅F beam.

473. Two wheel loads of 8 tons and 12 tons, respectively, are 10 ft. apart and roll across a 20-ft. span. Solve as in the preceding problem.

474. A wood beam 3 in. wide, 10 in. high, and 12 ft. long is simply supported at the ends. A concentrated load of 2 tons is at the center of the span. Compute the unit shearing stress at 1, 3, and 5 in. from the top of the beam.

475. A wood beam 4 in. wide, 14 in. high, and 18 ft. long is simply supported at the ends. A concentrated load of 3 tons is at the center of the span. Compute the unit shearing stress at 3, 5, and 7 in. from the bottom of the beam.

476. Compute the height of a wood floor joist $2\frac{1}{2}$ in. wide and 14 ft. long if it is to carry a load of 2800 lb. uniformly distributed over its entire span. $S = 1200$ lb./sq. in., $S_s = 100$ lb./sq. in.

477. A 15-in. 50-lb. American Standard I-beam, 20 ft. long, with end supports has a concentrated load of 10 tons 8 ft. from one end. Compute the maximum unit shearing stress by the (*a*) exact formula, (*b*) approximate formula.

478. Solve the preceding problem for a 12-in. 50-lb. WF beam.

479. A 26-ft. simple beam is supported at the right end and 6 ft. from the left end. At the left end is a 7200-lb. concentrated load, and 10 ft. from the left end is another 7200-lb. concentrated load. Over the right 10 ft. is a uniformly distributed load of 3600 lb. Design an economical wood beam of actual commercial size. $S = 1040$ lb./sq. in., $S_s = 80$ lb./sq. in.

480. A simple beam on end supports has a concentrated load P at the center of the span. E and I are constant. From the equation of the elastic curve, find the slope of the curve and its deflection at a distance $L/4$ from a support. Origin at a support.

481. Solve the preceding problem using a uniformly distributed load, $W = wL$, instead of the concentrated load.

482. A cantilever beam has a concentrated load at the free end. E and I are constants. From the equation of the elastic curve, find the slope of the elastic curve and its deflection at a distance $3/4L$ from the support. Origin at the support.

483. Solve the preceding problem using a uniformly distributed load, $W = wL$, instead of the concentrated load.

484. Solve Problem 480 by the Slope-deflection Method.

485. Solve Problem 480 by the Moment-area Method.

486. Solve Problem 480 by the Theorem of Three Moments Equation.

487. Solve Problem 481 by the Slope-deflection Method.

488. Solve Problem 481 by the Theorem of Three Moments Equation.

489. A simple beam on end supports is loaded at the quarter points with two equal concentrated loads. E and I are constants. Find the maximum deflection by the Slope-deflection Method.

490. Solve the preceding problem by the Moment-area Method.

491. Solve Problem 489 by the Theorem of Three Moments Equation.

492. A continuous beam rests on three supports at the same level. Each panel is 18 ft. long and has a concentrated load of 12 tons at 6 ft. from the end support. Compute sufficient data, draw the shear and moment diagrams, and determine the maximum shear and maximum moment.

493. Solve the preceding problem if the concentrated loads are replaced by 12 ton uniformly distributed loads extending from the free ends for 12 ft.

494. A continuous beam rests on three level supports. The left panel is 20 ft. long and has a uniform load of 9 tons over the 20 ft. The right panel is 16 ft. long and has a concentrated load of 7 tons at 7 ft. from the right pier. Draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

495. Considering its own weight, design an economical steel beam for the preceding problem, $S = 10$ tons/sq. in., (*a*) American Standard I-beam, (*b*) WF beam.

496. A continuous beam rests on three level supports. The left panel is 16 ft. and has 9 tons concentrated 4 ft. from the left end. The right panel is 18 ft. and has 6 tons uniformly distributed over its length. Compute sufficient data, draw the shear and moment diagrams, and determine the maximum shear and the maximum moment.

497. Considering its own weight, select an economical steel beam for the loads of the preceding problem, (*a*) American Standard I-beam, (*b*) WF beam. Safe flexure stress is 10 tons/sq. in.

498. A continuous beam on three level supports is 36 ft. long. The left panel is 20 ft. long and has a uniform load of 12 tons on the left 6 ft. The right panel is 16 ft. long and has a concentrated load of 6 tons placed 5 ft. from the right end. Compute sufficient data, draw the shear and moment diagrams, and find the maximum shear and the maximum moment.

499. Considering its own weight, design an economical steel beam for the loads of the preceding problem (*a*) American Standard I-beam, (*b*) WF beam. Safe bending stress is 10 tons/sq. in.

500. A continuous beam on three level supports is 31 ft. long. The left support is 3 ft. from the left end of the beam, the right support is at the right end, and the third support is 16 ft. from the right end of the beam. One ton is concentrated at the left end, 9 tons is 7 ft. from the left end, and 6 tons is uniformly distributed over the right panel. Design an economical wood beam ($2b = h$ approx.) for a bending stress of 1200 lb./sq. in.

501. Considering its own weight, design an economical steel beam for the loads of the preceding problem, using a flexure stress of 10 tons/sq. in. (*a*) American Standard I-beam, (*b*) WF beam.

502. Using the Theorem of Three Moments Equation, compute the deflection at the center of the left panel of Problem 500. $E = (10)^6$ lb./sq. in.

503. A continuous beam on four level supports is 40 ft. long. The two end panels are each 14 ft. long and have a uniformly distributed load of 10 tons over the 14 ft. The center panel supports a concentrated load of 5 tons at mid-span. Compute sufficient data, draw shear and moment diagrams, and determine the maximum shear and maximum moment.

504. A continuous beam on four level supports is 26 ft. long. The two end panels are each 8 ft. long and have a concentrated load of 8 tons at mid-span. The middle panel has a uniform load of 15 tons distributed over the 10 ft. Compute sufficient data, draw the shear and moment diagrams, and determine the maximum shear and moment.

505. In an elemental block subject to shear and uniaxial stress the maximum normal stress is 10,000 lb./sq. in. and the maximum shearing stress is 7000 lb./sq. in. Compute the simple normal and shearing unit stresses on the faces of the block.

506. In an elemental block subject to shear and uniaxial stress the maximum normal stress is 15,000 lb./sq. in. and the maximum shearing stress is 12,000 lb./sq. in. Compute the simple direct normal and shearing unit stresses on the faces of the block.

507. A 0.75- by 1.5-in. rectangular steel rod 3 ft. in length is used as a round-ended column. Compute the breaking load, using Euler's equation.

508. Compute the safe load for a 3- by 3- by $\frac{3}{8}$-in. angle 7 ft. long, used as a column. Use Eq. (24) or (25).

509. Compute the safe load for a 4- by 4- by $\frac{1}{4}$-in. angle 10 ft. long, used as a column. Use Eq. (24) or (25).

510. Compute the safe load for a 3- by 2- by $\frac{3}{8}$-in. angle 4 ft. long, used as a column. Use Eq. (24) or (25).

511. Compute the safe load of a 4- by 3- by $\frac{3}{8}$-in. angle 5 ft. long, used as a column. Use Eq. (24) or (25).

512. Compute the safe load for a 5- by 3- by $\frac{3}{8}$-in. angle 5 ft. long, used as a column. Use Eq. (24) or (25).

513. Compute the safe load for a 6- by 6-in. wood column 12 ft. long. [Eq. (27).]

514. Compute the safe load for a 6- by 6-in. wood column 12 ft. long, if it is laterally braced at mid-height. [Eq. (27).]

515. Compute the safe load for a 4- by 6-in. wood column 8 ft. long. [Eq. (27).]

516. Design an economical angle with unequal legs 8 ft. 04 in. long to support a safe load of 70,000 lb. [Eq. (24) or (25).]

517. Design an economical angle with unequal legs 8 ft. 04 in. long to support a safe load of 90,000 lb. [Eq. (24) or (25).]

518. Solve Problem 516 for an angle with equal legs.

519. Solve Problem 517 for an angle with equal legs.

520. Design an economical wood column of nominal commercial size 15 ft. long for a safe load of 45,000 lb. [Eq. (27).]

521. Design an economical wood column of nominal commercial size 15 ft. long for a safe load of 90,000 lb. [Eq. (27).]

522. Design an economical wood member of nominal commercial size 16 ft. long to support an axial thrust of 6000 lb. and an eccentric beam load of 3200 lb. placed 6 ft. from one end of the member. S = 1100 lb./sq. in. [Eq. (27).]

523. Design an economical wood member of nominal commercial size 10 ft. long to support an axial thrust of 12 tons and an eccentric beam load of 3 tons placed 7 ft. from one end of the member. S = 1100 lb./sq. in. [Eq. (27).]

524. Design a pair of economical American Standard channels 10 ft. long to support an axial thrust of 18 tons and an eccentric beam load of 6 tons placed 2 ft. from one end of the member. S = 9 tons/sq. in. [Eq. (24) or (25).]

525. Design a pair of economical American Standard channels 10 ft. long to support an axial thrust of 15 tons and an eccentric beam load of 5 tons placed 3 ft. from one end of the member. S = 9 tons/sq. in. [Eq. (24) or (25).]

526. Solve Problem 520 for the actual commercial size.

527. Solve Problem 521 for the actual commercial size.

528. Solve Problem 522 for the actual commercial size.

529. Solve Problem 523 for the actual commercial size.

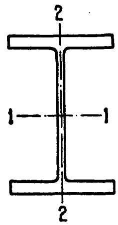
APPENDIX

REFERENCE TABLES

TABLE I.—AVERAGE HEAVINESS AND COEFFICIENT OF EXPANSION

Material	Weight, lb./ft.^{3}	Linear coefficient of thermal expansion per degree Fahrenheit η
Aluminum	168	0.000 0128
Brick:		
Common soft	120	
Good quality hard	144	0.000 0030
Masonry (ordinary)	120	
Concrete:		
Stone or gravel	150	0.000 0060
Cinder	110	
Copper	555	0.000 0093
Iron:		
Grey cast	450	0.000 0060
Wrought	480	0.000 0067
Steel	490	0.000 0065
Wood:		
Cedar and spruce	26	
Hemlock and soft pine	30	
Douglas fir and tamarack	36	
Ash and maple	38	
Southern yellow pine	40	0.000 0030
Hickory and oak	48	

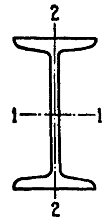
TABLE II.—WF SECTIONS*
Properties of sections



Depth of section, in.	Weight per foot, lb.	Area of section, in.²	Flange		Web thickness, in.	Axis 1-1			Axis 2-2		
			Width, in.	Thickness, in.		I, in.^{4}	Z, in.^{3}	k, in.	I, in.^{4}	Z, in.^{3}	k, in.
36.72	300	88.17	16.66	1.680	0.945	20290.2	1105.1	15.17	1225.2	147.1	3.73
36.00	240	70.60	16.50	1.320	0.790	15724.0	873.6	14.92	920.1	111.5	3.61
30.00	180	52.89	15.00	1.125	0.670	8328.2	555.2	12.55	585.6	78.1	3.33
24.00	100	29.43	12.00	0.775	0.468	2987.3	248.9	10.08	203.5	33.9	2.63
17.90	47	13.81	7.492	0.520	0.350	736.4	82.3	7.30	33.5	9.0	1.56
16.32	78	22.92	8.586	0.875	0.529	1042.6	127.8	6.74	87.5	20.4	1.95
14.00	87	25.56	14.500	0.688	0.420	966.9	138.1	6.15	349.7	48.2	3.70
12.00	28	8.23	6.500	0.420	0.240	213.5	35.6	5.09	17.5	5.4	1.46
10.50	72	21.18	10.170	0.808	0.510	420.7	80.1	4.46	141.8	27.9	2.59
10.38	66	19.41	10.117	0.748	0.457	382.5	73.7	4.44	129.2	25.5	2.58
10.00	49	14.40	10.000	0.558	0.340	272.9	54.6	4.35	93.0	18.6	2.54
10.22	29	8.53	5.799	0.500	0.289	157.3	30.8	4.29	15.2	5.2	1.34
9.90	21	6.19	5.750	0.340	0.240	106.3	21.5	4.14	9.7	3.4	1.25
8.00	31	9.12	8.000	0.433	0.288	109.7	27.4	3.47	37.0	9.2	2.01

* Of the many WF sections, Table II gives the properties of beams referred to in the text.

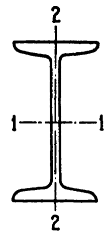
TABLE III.—PROPERTIES OF AMERICAN STANDARD BEAMS*



Depth of beam, in.	Weight per foot, lb.	Area of section, in.²	Width of flange, in.	Web thickness, in.	Axis 1-1 I, in.⁴	Axis 1-1 $Z = \frac{I}{c}$, in.³	Axis 1-1 k, in.	Axis 2-2 I, in.⁴	Axis 2-2 $Z = \frac{I}{c}$, in.³	Axis 2-2 k, in.
	120.0	35.13	8.048	0.798	3010.8	250.9	9.26	84.9	21.1	1.56
	115.0	33.67	7.987	0.737	2940.5	245.0	9.35	82.8	20.7	1.57
24	110.0	32.18	7.925	0.675	2869.1	239.1	9.44	80.6	20.3	1.58
	105.9	30.98	7.875	0.625	2811.5	234.3	9.53	78.9	20.0	1.60
	100.0	29.25	7.247	0.747	2371.8	197.6	9.05	48.4	13.4	1.29
	95.0	27.79	7.186	0.686	2301.5	191.8	9.08	47.0	13.0	1.30
24	90.0	26.30	7.124	0.624	2230.1	185.8	9.21	45.5	12.8	1.32
	85.0	24.84	7.063	0.563	2159.8	180.0	9.33	44.2	12.5	1.33
	79.9	23.33	7.000	0.500	2087.2	173.9	9.46	42.9	12.2	1.36
	100.0	29.20	7.273	0.873	1648.3	164.8	7.51	52.4	14.4	1.34
	95.0	27.74	7.200	0.800	1599.7	160.0	7.59	50.5	14.0	1.35
20	90.0	26.26	7.126	0.726	1550.3	155.0	7.68	48.7	13.7	1.36
	85.0	24.80	7.053	0.653	1501.7	150.2	7.78	47.0	13.3	1.38
	81.4	23.74	7.000	0.600	1466.3	146.6	7.86	45.8	13.1	1.39
	75.0	21.90	6.391	0.641	1263.5	126.3	7.60	30.1	9.4	1.17
20	70.0	20.42	6.317	0.567	1214.2	121.4	7.71	28.9	9.2	1.19
	65.4	19.08	6.250	0.500	1169.5	116.9	7.83	27.9	8.9	1.21
	70.0	20.46	6.251	0.711	917.5	101.9	6.70	24.5	7.8	1.09
	65.0	18.98	6.169	0.629	877.7	97.5	6.80	23.4	7.6	1.11
18	60.0	17.50	6.087	0.547	837.8	93.1	6.92	22.3	7.3	1.13
	54.7	15.94	6.000	0.460	795.5	88.4	7.07	21.2	7.1	1.15
	75.0	21.85	6.278	0.868	687.2	91.6	5.61	30.6	9.8	1.18
	70.0	20.38	6.180	0.770	659.6	87.9	5.69	28.8	9.3	1.19
15	65.0	18.91	6.082	0.672	632.1	84.3	5.78	27.2	8.9	1.20
	60.8	17.68	6.000	0.590	609.0	81.2	5.87	26.0	8.7	1.21
	55.0	16.06	5.738	0.648	508.7	67.8	5.63	17.0	5.9	1.03
	50.0	14.59	5.640	0.550	481.1	64.2	5.74	16.0	5.7	1.05
15	45.0	13.12	5.542	0.452	453.6	60.5	5.88	15.0	5.4	1.07
	42.9	12.49	5.500	0.410	441.8	58.9	5.95	14.6	5.3	1.08
	55.0	16.04	5.600	0.810	319.3	53.2	4.46	17.3	6.2	1.04
	50.0	14.57	5.477	0.687	301.6	50.3	4.55	16.0	5.8	1.05
12	45.0	13.10	5.355	0.565	284.1	47.3	4.66	14.8	5.5	1.06
	40.8	11.84	5.250	0.460	268.9	44.8	4.77	13.8	5.3	1.08

* From "A.I.S.C. Manual."

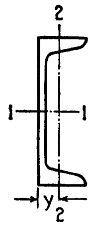
TABLE III.—PROPERTIES OF AMERICAN STANDARD BEAMS.*—(*Continued*)



Depth of beam, in.	Weight per foot, lb.	Area of section, in.²	Width of flange, in.	Web thickness, in.	Axis 1-1			Axis 2-2		
					I, in.⁴	$Z = \frac{I}{c}$, in.³	k, in.	I, in.⁴	$Z = \frac{I}{c}$, in.³	k, in.
12	35.0	10.20	5.078	0.428	227.0	37.8	4.72	10.0	3.9	0.99
	31.8	9.26	5.000	0.350	215.8	36.0	4.83	9.5	3.8	1.01
10	40.0	11.69	5.091	0.741	158.0	31.6	3.68	9.4	3.7	0.90
	35.0	10.22	4.944	0.594	145.8	29.2	3.78	8.5	3.4	0.91
	30.0	8.75	4.797	0.447	133.5	26.7	3.91	7.6	3.2	0.93
	25.4	7.38	4.660	0.310	122.1	24.4	4.07	6.9	3.0	0.97
8	25.5	7.43	4.262	0.532	68.1	17.0	3.03	4.7	2.2	0.80
	23.0	6.71	4.171	0.441	64.2	16.0	3.09	4.4	2.1	0.81
	20.5	5.97	4.079	0.349	60.2	15.1	3.18	4.0	2.0	0.82
	18.4	5.34	4.000	0.270	56.9	14.2	3.26	3.8	1.9	0.84
7	20.0	5.83	3.860	0.450	41.9	12.0	2.68	3.1	1.6	0.74
	17.5	5.09	3.755	0.345	38.9	11.1	2.77	2.9	1.6	0.76
	15.3	4.43	3.660	0.250	36.2	10.4	2.86	2.7	1.5	0.78
6	17.25	5.02	3.565	0.465	26.0	8.7	2.28	2.3	1.3	0.68
	14.75	4.29	3.443	0.343	23.8	7.9	2.36	2.1	1.2	0.69
	12.5	3.61	3.330	0.230	21.8	7.3	2.46	1.8	1.1	0.72
5	14.75	4.29	3.284	0.494	15.0	6.0	1.87	1.7	1.0	0.63
	12.25	3.56	3.137	0.347	13.5	5.4	1.95	1.4	0.91	0.63
	10.0	2.87	3.000	0.210	12.1	4.8	2.05	1.2	0.82	0.65
4	10.5	3.05	2.870	0.400	7.1	3.5	1.52	1.0	0.70	0.57
	9.5	2.76	2.796	0.326	6.7	3.3	1.56	0.91	0.65	0.58
	8.5	2.46	2.723	0.253	6.3	3.2	1.60	0.83	0.61	0.58
	7.7	2.21	2.660	0.190	6.0	3.0	1.64	0.77	0.58	0.59
3	7.5	2.17	2.509	0.349	2.9	1.9	1.15	0.59	0.47	0.52
	6.5	1.88	2.411	0.251	2.7	1.8	1.19	0.51	0.43	0.52
	5.7	1.64	2.330	0.170	2.5	1.7	1.23	0.46	0.40	0.53

* From "A.I.S.C. Manual."

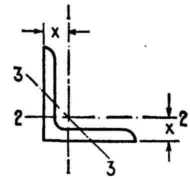
TABLE IV.—PROPERTIES OF AMERICAN STANDARD CHANNELS*



Depth of channel, in.	Weight per foot, lb.	Area of section, in.²	Width of flange, in.	Web thickness, in.	Axis 1-1			Axis 2-2			
					I, in.⁴	$Z = \frac{I}{c}$, in.³	k, in.	I, in.⁴	$Z = \frac{I}{c}$, in.³	k, in.	y, in.
	55.0	16.11	3.814	0.814	429.0	57.2	5.16	12.1	4.1	0.87	0.82
	50.0	14.64	3.716	0.716	401.4	53.6	5.24	11.2	3.8	0.87	0.80
	45.0	13.17	3.618	0.618	373.9	49.8	5.33	10.3	3.6	0.88	0.79
15	40.0	11.70	3.520	0.520	346.3	46.2	5.44	9.3	3.4	0.89	0.78
	35.0	10.23	3.422	0.422	318.7	42.5	5.58	8.4	3.2	0.91	0.79
	33.9	9.90	3.400	0.400	312.6	41.7	5.62	8.2	3.2	0.91	0.79
	40.0	11.73	3.415	0.755	196.5	32.8	4.09	6.6	2.5	0.75	0.72
	35.0	10.26	3.292	0.632	178.8	29.8	4.18	5.9	2.3	0.76	0.69
12	30.0	8.79	3.170	0.510	161.2	26.9	4.28	5.2	2.1	0.77	0.68
	25.0	7.32	3.047	0.387	143.5	23.9	4.43	4.5	1.9	0.79	0.68
	20.7	6.03	2.940	0.280	128.1	21.4	4.61	3.9	1.7	0.81	0.70
	35.0	10.27	3.180	0.820	115.2	23.0	3.34	4.6	1.9	0.67	0.69
	30.0	8.80	3.033	0.673	103.0	20.6	3.42	4.0	1.7	0.67	0.65
10	25.0	7.33	2.886	0.526	90.7	18.1	3.52	3.4	1.5	0.68	0.62
	20.0	5.86	2.739	0.379	78.5	15 7	3.66	2.8	1.3	0.70	0.61
	15.3	4.47	2.600	0.240	66.9	13.4	3.87	2.3	1.2	0.72	0.64
	25.0	7.33	2.812	0.612	70.5	15.7	3.10	3.0	1.4	0.64	0.61
	20.0	5.86	2.648	0.448	60.6	13.5	3.22	2.4	1.2	0.65	0.59
9	15.0	4.39	2.485	0.285	50.7	11.3	3.40	1.9	1.0	0.67	0.59
	13.4	3.89	2.430	0.230	47.3	10.5	3.49	1.8	0.97	0.67	0.61
	21.25	6.23	2.619	0.579	47.6	11.9	2.77	2.2	1.1	0.60	0.59
	18.75	5.49	2.527	0.487	43.7	10.9	2.82	2.0	1.0	0.60	0.57
8	16.25	4.76	2.435	0.395	39.8	9.9	2.89	1.8	0.94	0.61	0.56
	13.75	4.02	2.343	0.303	35.8	9.0	2.99	1.5	0.86	0.62	0.56
	11.5	3.36	2.260	0.220	32.3	8.1	3.10	1.3	0.79	0.63	0.58
	19.75	5.79	2.509	0.629	33.1	9.4	2.39	1.8	0.96	0.56	0.58
	17.25	5.05	2.404	0.524	30.1	8.6	2.44	1.6	0.86	0.56	0.55
7	14.75	4.32	2.299	0.419	27.1	7.7	2.51	1.4	0.79	0.57	0.53
	12.25	3.58	2.194	0.314	24.1	6.9	2.59	1.2	0.71	0.58	0.53
	9.8	2.85	2.090	0.210	21.1	6.0	2.72	0.98	0.63	0.59	0.55
	15.5	4.54	2.279	0.559	19.5	6.5	2.07	1.3	0.73	0.53	0.55
	13.0	3.81	2.157	0.437	17.3	5.8	2.13	1.1	0.65	0.53	0.52
6	10.5	3.07	2.034	0.314	15.1	5.0	2.22	0.87	0.57	0.53	0.50
	8.2	2.39	1.920	0.200	13.0	4.3	2.34	0.70	0.50	0.54	0.52
	11.5	3.36	2.032	0.472	10.4	4.1	1.76	0.82	0.54	0.49	0.51
5	9.0	2.63	1.885	0.325	8.8	3.5	1.83	0.64	0.45	0.49	0.48
	6.7	1.95	1.750	0.190	7.4	3.0	1.95	0.48	0.38	0.50	0.49
	7.25	2.12	1.720	0.320	4.5	2.3	1.47	0.44	0.35	0.46	0.46
4	6.25	1.82	1.647	0.247	4.1	2.1	1.50	0.38	0.32	0.45	0.46
	5.4	1.56	1.580	0.180	3.8	1.9	1.56	0.32	0.29	0.45	0.46
	6.0	1.75	1.596	0.356	2.1	1.4	1.08	0.31	0.27	0.42	0.46
3	5.0	1.46	1.498	0.258	1.8	1.2	1.12	0.25	0.24	0.41	0.44
	4.1	1.19	1.410	0.170	1.6	1.1	1.17	0.20	0.21	0.41	0.44

* From "A.I.S.C. Manual."

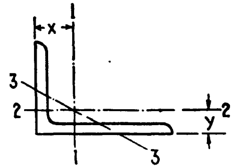
TABLE V.—PROPERTIES OF EQUAL ANGLES*



Size, in.	Thickness, in.	Weight per foot, lb.	Area of section, in.²	Axis 1-1 and axis 2-2: I, in.⁴	k, in.	S, in.³	x, in.	Axis 3-3: k min., in.
6 × 6	1	37.4	11.00	35.5	1.80	8.6	1.86	1.17
	15/16	35.3	10.37	33.7	1.80	8.1	1.84	1.17
	7/8	33.1	9.73	31.9	1.81	7.6	1.82	1.17
	13/16	31.0	9.09	30.1	1.82	7.2	1.80	1.17
	3/4	28.7	8.44	28.2	1.83	6.7	1.78	1.17
	11/16	26.5	7.78	26.2	1.83	6.2	1.75	1.17
	5/8	24.2	7.11	24.2	1.84	5.7	1.73	1.18
	9/16	21.9	6.43	22.1	1.85	5.1	1.71	1.18
	1/2	19.6	5.75	19.9	1.86	4.6	1.68	1.18
	7/16	17.2	5.06	17.7	1.87	4.1	1.66	1.19
	3/8	14.9	4.36	15.4	1.88	3.5	1.64	1.19
5 × 5	1	30.6	9.00	19.6	1.48	5.8	1.61	0.97
	15/16	28.9	8.50	18.7	1.48	5.5	1.59	0.97
	7/8	27.2	7.98	17.8	1.49	5.2	1.57	0.97
	13/16	25.4	7.46	16.8	1.50	4.9	1.55	0.97
	3/4	23.6	6.94	15.7	1.51	4.5	1.52	0.97
	11/16	21.8	6.40	14.7	1.51	4.2	1.50	0.98
	5/8	20.0	5.86	13.6	1.52	3.9	1.48	0.98
	9/16	18.1	5.31	12.4	1.53	3.5	1.46	0.98
	1/2	16.2	4.75	11.3	1.54	3.2	1.43	0.98
	7/16	14.3	4.18	10.0	1.55	2.8	1.41	0.98
	3/8	12.3	3.61	8.7	1.56	2.4	1.39	0.99
4 × 4	3/4	18.5	5.44	7.7	1.19	2.8	1.27	0.78
	11/16	17.1	5.03	7.2	1.19	2.6	1.25	0.78
	5/8	15.7	4.61	6.7	1.20	2.4	1.23	0.78
	9/16	14.3	4.18	6.1	1.21	2.2	1.21	0.78
	1/2	12.8	3.75	5.6	1.22	2.0	1.18	0.78
	7/16	11.3	3.31	5.0	1.23	1.8	1.16	0.78
	3/8	9.8	2.86	4.4	1.23	1.5	1.14	0.79
	5/16	8.2	2.40	3.7	1.24	1.3	1.12	0.79
	1/4	6.6	1.94	3.0	1.25	1.1	1.09	0.80
3½ × 3½	3/4	16.0	4.69	5.0	1.03	2.1	1.15	0.68
	11/16	14.8	4.34	4.7	1.04	2.0	1.12	0.68
	5/8	13.6	3.98	4.3	1.04	1.8	1.10	0.68
	9/16	12.4	3.62	4.0	1.05	1.7	1.08	0.68
	1/2	11.1	3.25	3.6	1.06	1.5	1.06	0.68
	7/16	9.8	2.87	3.3	1.07	1.3	1.04	0.68
	3/8	8.5	2.48	2.9	1.07	1.2	1.01	0.69
	5/16	7.2	2.09	2.5	1.08	0.98	0.99	0.69
	1/4	5.8	1.69	2.0	1.09	0.79	0.97	0.69
3 × 3	5/8	11.5	3.36	2.6	0.88	1.3	0.98	0.58
	9/16	10.4	3.06	2.4	0.89	1.2	0.95	0.58
	1/2	9.4	2.75	2.2	0.90	1.1	0.93	0.58
	7/16	8.3	2.43	2.0	0.91	0.95	0.91	0.58
	3/8	7.2	2.11	1.8	0.91	0.83	0.89	0.58
	5/16	6.1	1.78	1.5	0.92	0.71	0.87	0.59
	1/4	4.9	1.44	1.2	0.93	0.58	0.84	0.59
2½ × 2½	1/2	7.7	2.25	1.2	0.74	0.72	0.81	0.49
	7/16	6.8	2.00	1.1	0.75	0.65	0.78	0.49
	3/8	5.9	1.73	0.98	0.75	0.57	0.76	0.49
	5/16	5.0	1.47	0.85	0.76	0.48	0.74	0.49
	1/4	4.1	1.19	0.70	0.77	0.39	0.72	0.49
	3/16	3.07	0.90	0.55	0.78	0.30	0.69	0.49

The foregoing table is a partial listing only. Other sizes are available.
* From "A.I.S.C. Manual."

TABLE VI.—PROPERTIES OF UNEQUAL ANGLES*



Size, in.	Thick-ness, in.	Weight per foot, lb.	Area of section, in.^{2}	Axis 1-1				Axis 2-2				Axis 3-3
				I, in.^{4}	k, in.	$Z = \frac{I}{c}$, in.^{3}	x, in.	I, in.^{4}	k, in.	$Z = \frac{I}{c}$, in.^{3}	y, in.	k min., in.
6×4	1	30.6	9.00	30.8	1.85	8.0	2.17	10.8	1.09	3.8	1.17	0.86
	$\frac{15}{16}$	28.9	8.50	29.3	1.86	7.6	2.14	10.3	1.10	3.6	1.14	0.86
	$\frac{7}{8}$	27.2	7.98	27.7	1.86	7.2	2.12	9.8	1.11	3.4	1.12	0.86
	$\frac{13}{16}$	25.4	7.47	26.2	1.87	6.7	2.10	9.2	1.11	3.2	1.10	0.86
	$\frac{3}{4}$	23.6	6.94	24.5	1.88	6.3	2.08	8.7	1.12	3.0	1.08	0.86
	$\frac{11}{16}$	21.8	6.40	22.8	1.89	5.8	2.06	8.1	1.13	2.8	1.06	0.86
	$\frac{5}{8}$	20.0	5.86	21.1	1.90	5.3	2.03	7.5	1.13	2.5	1.03	0.86
	$\frac{9}{16}$	18.1	5.31	19.3	1.90	4.8	2.01	6.9	1.14	2.3	1.01	0.87
	$\frac{1}{2}$	16.2	4.75	17.4	1.91	4.3	1.99	6.3	1.15	2.1	0.99	0.87
	$\frac{7}{16}$	14.3	4.18	15.5	1.92	3.8	1.96	5.6	1.16	1.9	0.96	0.87
	$\frac{3}{8}$	12.3	3.61	13.5	1.93	3.3	1.94	4.9	1.17	1.6	0.94	0.88
5×3½	$\frac{3}{4}$	19.8	5.81	13.9	1.55	4.3	1.75	5.6	0.98	2.2	1.00	0.75
	$\frac{11}{16}$	18.3	5.37	13.0	1.56	4.0	1.72	5.2	0.98	2.1	0.97	0.75
	$\frac{5}{8}$	16.8	4.92	12.0	1.56	3.7	1.70	4.8	0.99	1.9	0.95	0.75
	$\frac{9}{16}$	15.2	4.47	11.0	1.57	3.3	1.68	4.5	1.00	1.7	0.93	0.75
	$\frac{1}{2}$	13.6	4.00	10.0	1.58	3.0	1.66	4.1	1.01	1.6	0.91	0.75
	$\frac{7}{16}$	12.0	3.53	8.9	1.59	2.6	1.63	3.6	1.01	1.4	0.88	0.76
	$\frac{3}{8}$	10.4	3.05	7.8	1.60	2.3	1.61	3.2	1.02	1.2	0.86	0.76
	$\frac{5}{16}$	8.7	2.56	6.6	1.61	1.9	1.59	2.7	1.03	1.0	0.84	0.77
4×3	$\frac{3}{4}$	16.0	4.69	6.9	1.22	2.7	1.42	3.3	0.84	1.6	0.92	0.64
	$\frac{11}{16}$	14.8	4.34	6.5	1.22	2.5	1.39	3.1	0.84	1.5	0.89	0.64
	$\frac{5}{8}$	13.6	3.98	6.0	1.23	2.3	1.37	2.9	0.85	1.4	0.87	0.64
	$\frac{9}{16}$	12.4	3.62	5.6	1.24	2.1	1.35	2.7	0.86	1.2	0.85	0.64
	$\frac{1}{2}$	11.1	3.25	5.1	1.25	1.9	1.33	2.4	0.86	1.1	0.83	0.64
	$\frac{7}{16}$	9.8	2.87	4.5	1.25	1.7	1.30	2.2	0.87	1.0	0.80	0.64
	$\frac{3}{8}$	8.5	2.48	4.0	1.26	1.5	1.28	1.9	0.88	0.87	0.78	0.64
	$\frac{5}{16}$	7.2	2.09	3.4	1.27	1.2	1.26	1.7	0.89	0.73	0.76	0.65
	$\frac{1}{4}$	5.8	1.69	2.8	1.28	1.0	1.24	1.4	0.90	0.60	0.74	0.65
3×2½	$\frac{1}{2}$	8.5	2.50	2.1	0.91	1.0	1.00	1.3	0.72	0.74	0.75	0.52
	$\frac{7}{16}$	7.6	2.21	1.9	0.92	0.93	0.98	1.2	0.73	0.66	0.73	0.52
	$\frac{3}{8}$	6.6	1.92	1.7	0.93	0.81	0.96	1.0	0.74	0.58	0.71	0.52
	$\frac{5}{16}$	5.6	1.62	1.4	0.94	0.69	0.93	0.90	0.74	0.49	0.68	0.53
	$\frac{1}{4}$	4.5	1.31	1.2	0.95	0.56	0.91	0.74	0.75	0.40	0.66	0.53
2½×2	$\frac{1}{2}$	6.8	2.00	1.1	0.75	0.70	0.88	0.64	0.56	0.46	0.63	0.42
	$\frac{7}{16}$	6.1	1.78	1.0	0.76	0.62	0.85	0.58	0.57	0.41	0.60	0.42
	$\frac{3}{8}$	5.3	1.55	0.91	0.77	0.55	0.83	0.51	0.58	0.36	0.58	0.42
	$\frac{5}{16}$	4.5	1.31	0.79	0.78	0.47	0.81	0.45	0.58	0.31	0.56	0.42
	$\frac{1}{4}$	3.62	1.06	0.65	0.78	0.38	0.79	0.37	0.59	0.25	0.54	0.42
	$\frac{3}{16}$	2.75	0.81	0.51	0.79	0.29	0.76	0.29	0.60	0.20	0.51	0.43

The foregoing table is a partial listing only. Many other sizes are available.
* From "A.I.S.C. Manual."

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